Proceedings

Tenth Annual Workshop

in Geometric Topology

Hosted by Oregon State University
and held at
Corvallis, Oregon and Newport, Oregon
June 10--12, 1993

The Tenth Annual Workshop in Geometric Topology was hosted by Oregon State University and was held at Corvallis, Oregon and Newport, Oregon on June 10–12, 1993. The participants were:

Fredric Ancel	University of Wisconsin - Milwaukee
Robert Andersen	University of Wisconsin - Eau Claire
Mladen Bestvina	UCLA
William Bogley	Oregon State University
John Bryant	Florida State University
Robert Daverman	University of Tennessee
Tadek Dobrowolski	University of Oklahoma
Robert Edwards	UCLA
Paul Fabel	University of Texas at Austin
Mark Feighn	Rutgers University
Eric Freden	Brigham Young University
Dennis Garity	Oregon State University
Craig Guilbault	University of Wisconsin - Milwaukee
Isa Jubran	Oregon State University
Patricia Hale	Oregon State University
John Hempel	Rice University
James Henderson	The Colorado College
Kiwon Nam	Oregon State University
David Radcliffe	University of Wisconsin - Milwaukee
Richard Schori	Oregon State University
Frank Shaw	University of Minnesota
Matthew Timm	Bradley University
Frederick Tinsley	The Colorado College
Paul Thurston	University of Tennessee
Gerard Venema	Calvin College
John Walsh	University of California - Riverside
David Wright	Brigham Young University

These proceedings contain a summary of the three one-hour talks delivered by the principal speaker, John Bryant. Summaries of talks given by some of the other participants are also included. The success of the workshop was helped by generous funding from Oregon State University and the National Science Foundation.

Editors: Dennis Garity
Fredric Ancel
Craig Guilbault
James Henderson
Frederick Tinsley
David Wright

Table of Contents

Counterexamples to the Resolution Conjecture

John L. Bryant

(Introductory notes prepared by Dennis Garity)

John Bryant gave a series of three talks on Counterexamples to the Resolution Conjecture at the Tenth Annual Workshop in Geometric Topology. A summary of his talks begins on the next page of these proceedings.

His talks represent joint work with Washington Mio, Steve Ferry and Shmuel Weinberger. For background terminology, additional references and a history of the problem, the reader is referred to the following two papers and a review of the second paper which appeared in Mathematical Reviews.

J. Bryant, S. Ferry, W. Mio, and S. Weinberger, *Topology of Homology Manifolds*, Bulletin of the American Mathematical society, Volume 28, Number 2, April 1993, pages 324-328.

Bryant, J.; Ferry, S.; Mio, W.; Weinberger, S. Topology of homology manifolds. Ann. of Math. (2) 143 (1996), no. 3, 435-467.

A. Ranicki, A Review of "Topology of homology manifolds" by J. Bryant, S. Ferry, W. Mio, and S. Weinberger, Mathematical Reviews 97b:57017.

Counterexamples to the Resolution Conjecture

John L. Bryant

Introduction

Definition. A *resolution* of a generalized n-manifold X is a cell-like map $f: M \to X$ where M is a topological n-manifold.

The Resolution Conjecture. Every generalized manifold X is resolvable.

Theorem. (Quinn) If X is a connected generalized n-manifold, $n \geq 5$, then there is at most one integral obstruction $\sigma(X) \in \mathbb{Z}$ to resolving X (a local signature). Moreover, this obstruction has a local character, i.e., it can be detected in any open set.

Theorem (Bryant, Ferry, Mio, Weinberger) If M is a simply-connected, closed manifold of dimension ≥ 6 and σ is an integer, then there is a generalized manifold X homotopy equivalent to M such that $\sigma(X) = \sigma$.

The construction of such examples depends upon controlled surgery theory as developed by Ferry and Pedersen. The theory has been developed in two essentially equivalent forms: bounded surgery theory and ϵ-surgery theory. We shall outline the construction of an example homotopy equivalent to S^n using ϵ-surgery.

ϵ-Surgery à la Ferry-Pedersen

Suppose that X and B are finite complexes. A map $p: X \to B$ is a UV^1 map if p is onto and each point-inverses. This means, essentially, that p has the approximate lifting property for maps of 2-dimensional polyhedra into B. X is an *ϵ-Poincaré space over B* if X is triangulated so that $\operatorname{diam} p(\sigma) \ll \epsilon$ for each simplex σ of X and there is a class $y \in H_n(X)$ such that $y \cap \_: C^*(X) \to C_{n-*}(X)$ is an ϵ-chain homotopy equivalence. An ϵ-surgery problem over B is a degree one normal map $\phi: W^n \to X$. The "problem" is to find a normal cobordism of ϕ to and ϵ-homotopy equivalence $\phi': W' \to X$ over B. Two problems $\phi_1: W_1^n \to X$ and $\phi_2: W_2^n \to X$ are equivalent if there is a normal bordism $F: (P; W_1, W_2) \to X$ extending ϕ_1 and ϕ_2. Given a UV^1 map $p: N \to B$, $S'_\epsilon\begin{pmatrix} N \\ \downarrow p \\ B \end{pmatrix}$ denotes the set whose elements are represented by $p^{-1}(\epsilon)$-homotopy equivalences $f: M \to N$. $f: M \to N$ and $f': M' \to N$ are equivalent if there is a homeomorphism $h: M \to M'$ such that the diagram

$$\begin{array}{ccc} M & & \\ h\downarrow & \overset{f}{\searrow} & \\ M' & \xrightarrow{f'} & N \\ & & p\downarrow \\ & & B \end{array}$$

ϵ-commutes over B.

<u>Existence Theorem</u>. (Ferry-Pedersen) Given B and $n \geq 5$, there exist $\epsilon_0 > 0$ and $T > 0$ such that, for $0 < \epsilon < \epsilon_0$, if ϕ is an ϵ-surgery problem over B, then there is a well-defined obstruction $\sigma \in H_n(B, \mathbf{L})$ that vanishes if, and only if, ϕ is normally bordant to a $T\epsilon$-homotopy equivalence over B.

$H_n(B, \mathbf{L})$ is the generalized homology of B with coefficients in the periodic simply connected surgery $\mathbf{L}$-spectrum:

$$L_n = \begin{cases} \mathbb{Z}, & \text{if } n \equiv 0 \pmod 4; \\ \mathbb{Z}/2, & \text{if } n \equiv 2 \pmod 4; \\ 0, & \text{otherwise.} \end{cases}$$

<u>ϵ-Surgery Exact Sequence</u>. (Ferry-Pedersen) Given B and $n \geq 5$, there exist $\epsilon_0 > 0$ and $T > 0$, depending on B and n, such that, for $0 < \epsilon < \epsilon_0$, the following sequence is exact:

$$\cdots \longrightarrow H_{n+1}(B; \mathbf{L}) \longrightarrow \mathcal{S}_\epsilon \left(\begin{smallmatrix} N \\ \downarrow p \\ B \end{smallmatrix} \right) \longrightarrow [N, G/Top] \longrightarrow H_n(B; \mathbf{L}),$$

where $\mathcal{S}_\epsilon \left(\begin{smallmatrix} N \\ \downarrow p \\ B \end{smallmatrix} \right) = im\Big(\mathcal{S}'_\epsilon \left(\begin{smallmatrix} N \\ \downarrow p \\ B \end{smallmatrix} \right) \to \mathcal{S}'_{T\epsilon} \left(\begin{smallmatrix} N \\ \downarrow p \\ B \end{smallmatrix} \right)\Big)$.

Main Results

<u>Theorem</u>. (Bryant, Ferry, Mio, Weinberger) If $n \geq 6$ and $\sigma \in \mathbb{Z}$, then there is a generalized n-manifold X homotopy equivalent to S^n with $\sigma(X) = \sigma$.

Sketch of Proof. Choose a fine triangulation of S^n, let N_0 denote the boundary of a regular neighborhood C_0 of its 2-skeleton, and let D_0 denote the closure of the complement of C_0. Results of Bestvina-Walsh-Ferry guarantee that the identity map on S^n is close homotopically to a map $p: S^n \to S^n$ such that the restriction of p to each of C_0, D_0, and N_0 is a UV^1 map onto S^n. Compare the controlled surgery exact sequence of N_0 with control map $p: N_0 \to S^n$ to the uncontrolled sequence via the "forget control" map; i.e., the constant map $S^n \to pt$:

$$\begin{array}{ccccccc}
 & & L_0 \oplus L_n & & & & \\
 & & \| & & & & \\
\cdots \to [\Sigma N_0, G/Top] & \to & H_n(S^n; \mathbf{L}) & \to & \mathcal{S}_\epsilon\left(\begin{smallmatrix} N \\ \downarrow p \\ B \end{smallmatrix}\right) & \to & [N_0, G/Top] \\
\downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\
\cdots \to [\Sigma N_0, G/Top] & \to & L_n & \to & \mathcal{S}(N_0) & \to & [N_0, G/Top]
\end{array}$$

Choose σ in the L_0-component of $H_n(S^n; \mathbf{L})$. (Recall: $L_0 \cong \mathbb{Z}$.) By the controlled version of Wall's Realization Theorem there is a degree one normal map $F: (W; N_0, N_1) \to (N_0 \times I; N_0 \times 0, N_0 \times 1)$ such that

(a) $F|N_0 =$ identity,

(b) $f_0 = proj \circ F|N_1: N_1 \to N_0$ is a $p^{-1}(\epsilon)$ equivalence over S^n, and

(c) the obstruction to doing surgery on W, rel boundary, to get an ϵ-homotopy equivalence over S^n is σ.

As an uncontrolled surgery problem, however, the obstruction is 0, hence we may assume (after doing surgery) that W is (uncontrolled) homotopy equivalent to $N_0 \times I$. By the h-cobordism theorem, W is homeomorphic to $N_0 \times I$. Cut S^n open along N_0, and glue C_0 and D_0 together along N_0 using f, forming the Poincaré complex X_0, homotopy equivalent to S^n via a map $p_0: X_0 \to S^n$. We may assume that p_0 is a UV^1 map, so that $(p_0)^*: H_n(X_0; \mathbf{L}) \to H_n(S^n; \mathbf{L})$ identifies surgery obstructions. Hence, we may consider $\sigma \in H_n(X_0; \mathbf{L})$.

Perform the construction on a finer triangulation of S^n obtaining a Poincaré complex $X_1' = C_1 \cup_{f_1} D_1, f_1: N_1 \to N_1$, a fine homotopy equivalence over X_0 with controlled obstruction σ, and (UV^1) homotopy equivalence $q_1: X_1' \to S^n$. Composing q_1 and a homotopy inverse to p_0 gives a map $p_1': X_1' \to X_0$. Considered as a controlled surgery problem over S^n, the obstruction to doing surgery on X_1' to get a fine homotopy equivalence is 0. Thus we may do surgery (away from C_1) to get a fine homotopy equivalence $p_1: X_1 \to X_0$ over S^n. The size of the homotopy equivalence is limited by the size of the Poincaré duality of X_0. In general, we cut and paste S^n to build finer and finer homotopy equivalences $p_i: X_i \to X_{i-1}$ over X_{i-2} ($X_{-1} = S^n$) of finer and finer Poincaré complexes. X_{i+1} is constructed in $\mathbf{R}^\ell$, for ℓ large, inside a regular neighborhood of W_i of X_i. With sufficient care in the construction, we obtain the desired space X as the intersection of the W_i's.∎

Remark 1. One may also construct an example using bounded surgery theory of Ferry-Pedersen, using the open cone on S^n as (initial) control space.

over X; hence, P may be chosen to be an arbitrarily fine Poincaré duality space over X. ∎

The construction of Y is now obtained as follows. Get a sequence $\{U_i \supseteq P_i \supseteq K_i\}$ as above with $\bigcap_{i=1}^{\infty} U_i = X$ and ϵ_i-equivalences $P_i \to X$. Use controlled surgery to get a sequence $\{W_i \supseteq Q_i \supseteq L_i\}$, starting with (U_1, P_1, K_1), and homeomorphisms $h_i \colon \mathbf{R}^\ell \to \mathbf{R}^\ell$ taking (W_i, Q_i, L_i) to (U_i, P_i, K_i) such that

(a) $W_1 \supseteq W_2 \supseteq \cdots$,

(b) $L_1 \subseteq L_2 \subseteq \cdots$,

(c) Q_{i+1} is δ_i-equivalent to Q_i over Q_{i-1}, and

(d) h_i converges to a cell-like map $\Phi \colon \mathbf{R}^\ell \to \mathbf{R}^\ell$ that is one-to-one over $\mathbf{R}^\ell - X$.

We set $Y = \bigcap_{i=1}^{\infty} W_i$ and $\phi = \Phi | Y$.

Remark 2. The construction can clearly be carried out with S^n replaced by an arbitrary simply connected manifold M.

Examples with the DDP

A metric space X has the DDP if maps $f, g: D^2 \to X$ of the 2-disk D^2 into X can be approximated by maps $f', g': D^2 \to X$ with disjoint images.

Theorem. (Bryant, Ferry, Mio, Weinberger) Given a generalized n-manifold X, $n \geq 6$, there is a generalized n-manifold Y with the DDP and a cell-like map $\phi: Y \to X$.

Sketch of proof. The proof is based upon the fact that the second derived subdivision of a triangulation T of a PL n-manifold N contains both the 2-skeleton and the dual 2-skeleton of T. If $n \geq 5$, then two maps $f, g: D^2 \to N$ are homotopic to maps $f', g': D^2 \to N$ where f' maps D^2 into the 2-skeleton of T and g' maps into the dual 2-skeleton, thus having disjoint images. The size of the homotopy is determined by the mesh of T.

Lemma. Suppose X is a generalized n-manifold, $n \geq 6$, $\epsilon > 0$ and $\eta > 0$. Then there is an n-dimensional η-Poincaré complex P over X that is ϵ-homotopy equivalent to X and contains the 2-skeleton of a triangulation of a mapping cylinder neighborhood of X.

Proof. Embed X as a $1-LCC$ subset of $\mathbf{R}^\ell$, for large ℓ, and let U be a mapping cylinder neighborhood of X in $\mathbf{R}^\ell$ with mapping cylinder retraction $\gamma: U \to X$. Choose a fine triangulation T of U. An argument, using Quinn's solution to Connell-Hollingsworth's geometric groups conjectures, shows that there is a small collapse of U onto an n-dimensional complex $Q \subseteq W$ that contains the 2-skeleton K of T. Q is an ϵ'-Poincaré complex over X, where ϵ' depends on the size of the retraction γ. By results of Ferry-Pedersen, there is a degree one normal map $g: M \to X$ of an n-manifold M onto X, which, after elementary surgeries, can be assumed to be δ-1-connected over X. After approximating g by embedding of M into $\mathbf{R}^\ell$, we can get an embedding of K into M that is close to the inclusion of K in $\mathbf{R}^\ell$. Let σ be the resolution obstruction of X and let N be the boundary of a regular neighborhood C of K in M. Use Wall's Realization Theorem to realize σ by a degree one normal map $F: (V; N, N') \to (N \times I, N \times 0, N \times 1)$ with $f = proj \circ F: N' \to N$ an η'-homotopy equivalence over X. Form the Poincaré complex $Q' = C \cup_f V \cup_{id_N} \overline{M - C}$ and get an ϵ-surgery problem $Q' \to Q$ over X that has obstruction 0. Thus, we may do surgery on Q' away from C to get an ϵ''-equivalence $P \to Q$ over X; hence, and ϵ-equivalence $P \to X$, assuming appropriate controls have been imposed so far. Squeezing theorems of Ferry-Pedersen imply that f may be chosen to be an arbitrarily fine homotopy equivalence

Homotopic Maps to S^1 Have Homeomorphic Mapping Swirls, and Consequence for Pseudo-Spines of 4-Manifolds

by

Fredric D. Ancel and Craig R. Guilbault

We call a compact contractible n-manifold a *homotopy n-ball.* A subset X of the interior of a manifold M is called a *(topological) spine* of M if M is homeomorphic to the mapping cylinder of a map from ∂M to X. X is called a *pseudo-spine* of M if M–X is homeomorphic to $\partial M\times[0,\infty)$.

In [1] it is proved that for $n\geq 5$, every homotopy n-ball has a wild arc spine. It is observed, however, that in general homotopy 4-balls don't have arc spines. In fact, a homotopy 4-ball with an arc spine must be either a 4-ball or the cone over a non-trivial homotopy 3-sphere (if one exists). Thus, a homotopy 4-ball with a non-simply connected boundary can't have an arc spine.

The Mazur 4-manifold [6] is a homotopy 4-ball with a non-simply connected boundary. It is a celebrated consequence of [5] and [3] that the Mazur 4-manifold has an arc pseudo-spine.

The naively optimistic conjecture motivating this paper is: every homotopy 4-ball has an arc pseudo-spine. We will reinterpret and generalize the method of [5] through the introduction of the mapping swirl construction. We will prove several theorems about mapping swirls which allow us to produce canonical pseudo-spines for a special class of compact 4-manifolds which includes the Mazur 4-manifold. (This class of compact 4-manifolds consists of all those obtained by attaching finitely many 2-handles to $B^3\times S^1$.) We will then speculate about the possibility of finding simple pseudo-spines for all compact 4-manifolds and, in particular, for homotopy 4-balls.

1. Motivation: the Psuedo-Spine of the Mazur 4-Manifold

We briefly sketch the proof that the Mazur 4-manifold has an arc pseudo-spine to motivate subsequent developments. The Mazur 4-manifold M^4 is obtained by attaching a 2-handle to $B^3\times S^1$ along a curve J in $\partial B^3\times S^1$. Corresponding to this description, one finds that M^4 has a "dunce hat" spine which is the union of the disk D^2 which is the core of the 2-handle and the mapping cylinder $\mathrm{Cyl}(\pi|J)$ of the restriction to J of the natural projection $\pi : B^3\times S^1 \to \{0\}\times S^1$ of $B^3\times S^1$ onto its core. (See Figure 1.) Now in the dunce hat spine, replace the mapping cylinder $\mathrm{Cyl}(\pi|J)$ by the "mapping swirl" $\mathrm{Swl}(\pi|J)$ in which the fiber emanating from a point p on J, instead of running straight from p to its image $\pi(p)$ in $\{0\}\times S^1$, spirals infinitely in the S^1-direction in $B^3\times S^1$ as it approaches $\{0\}\times S^1$. (See Figure 2.) The resulting object $D^2\cup\mathrm{Swl}(\pi|J)$ is, according to [5], a pseudo-spine of

M^4 and (remarkably) a topological disk wildly embedded in M^4. (See Figure 2.) Thus, M^4 has a disk pseudo-spine. The result of [3] then allows us to "squeeze" this disk to an arc to conclude that M^4 has an arc pseudo-spine.

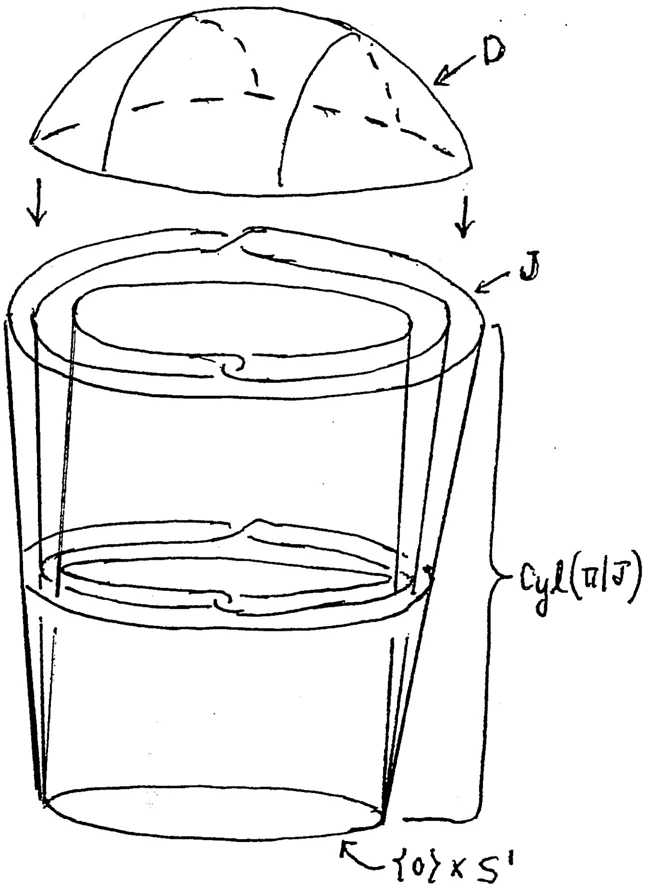


Figure 1

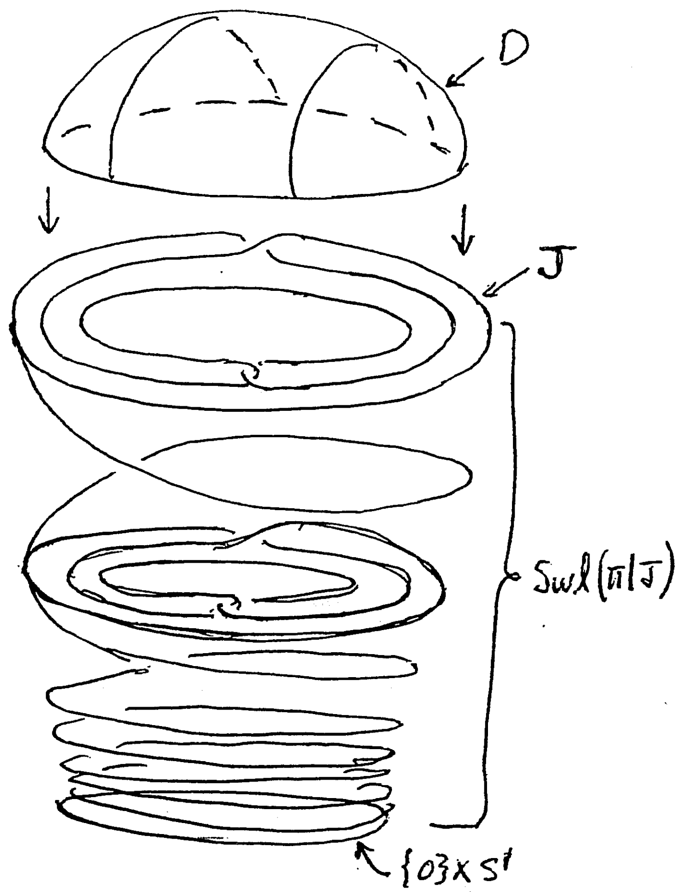


Figure 2

2. The Mapping Swirl Construction: Definitions and Statements of Results

We will now give a formal definition of the object which we called the "mapping swirl" in the preceding section. For a space X, we define the *cone* on X, denoted C(X), to be the quotient space $[0,\infty]\times X/\{\infty\}\times X$. (It is convenient for our purposes to make this slightly non-standard definition of C(X) instead of the more usual $[0,1]\times X/\{1\}\times X$.) For $(t,x) \in [0,\infty]\times X$, we let tx denote the corresponding point of C(X); and we let ∞ denote the point of C(X) corresponding to $\{\infty\}\times X$. We similarly define the *suspension* of X, denoted $\Sigma(X)$, to be the quotient space $[-\infty,\infty]\times X/\{\{-\infty\}\times X, \{\infty\}\times X\}$; and for $(t,x) \in [-\infty,\infty]\times X$, we let tx denote the corresponding point of $\Sigma(X)$; and we let $\pm\infty$ denote the points of $\Sigma(X)$ corresponding to $\{\pm\infty\}\times X$.

Let $f : X \to Y$ be a map. To motivate our upcoming definition of Swl(f), we observe that Cyl(f), the mapping cylinder of f, naturally embeds in $C(X)\times Y$. We regard Cyl(f) as the quotient space $X\times[0,\infty]\cup Y/\sim$ where $\sim$ identifies (x,∞) with $f(x)$ for $x \in X$. To embed Cyl(f) in $C(X)\times Y$, we identify the equivalence class $[(x,t)] \in \mathrm{Cyl}(f)$ of the point (x,t) $\in X\times[0,\infty)$ with the point $(tx,f(x)) \in C(X)\times Y$, and we identify the equivalence class $[y] \in$ Cyl(f) of the point $y \in Y$ with the point $(\infty,y) \in C(X)\times Y$. In other words, Cyl(f) is identified with the subset

$$\{ (tx,f(x)) \in C(X)\times Y : (t,x) \in [0,\infty)\times X \} \cup (\{\infty\}\times Y)$$

of $C(X)\times Y$. A similar observation reveals that the double mapping cylinder of f, DblCyl(f), naturally embeds in $\Sigma(X)\times Y$ as the subset

$$\{ (tx,f(x)) \in \Sigma(X)\times Y : (t,x) \in (-\infty,\infty)\times X \} \cup (\{-\infty\}\times Y) \cup (\{\infty\}\times Y).$$

Now consider a map $f : X \to S^1$. Define the *mapping swirl* of f, denoted Swl(f) to be the subset

$$\{ (tx,e^{2\pi i t}f(x)) \in C(X)\times S^1 : (t,x) \in [0,\infty)\times X \} \cup (\{\infty\}\times S^1)$$

of $C(X)\times S^1$. Similarly we define the *double mapping swirl* of f, denoted DblSwl(f) to be the subset

$$\{ (tx,e^{2\pi i t}f(x)) \in \Sigma(X)\times S^1 : (t,x) \in (-\infty,\infty)\times X \} \cup (\{-\infty,\infty\}\times S^1).$$

For each integer n, let $\phi_n : S^1 \to S^1$ denote the map $\phi_n(z) = z^n$; and let X(n) denote the adjunction space $B^2\cup_{\phi_n}S^1$. Thus, $X(\pm1)$ is a disk, and $X(\pm2)$ is a projective plane. For integers $n_1, n_2, \cdots, n_k$, let $X(n_1, n_2, \cdots, n_k)$ denote the adjunction space $(B^2\times\{1,2,\cdots,k\})\cup_\Phi S^1$ where $\Phi : S^1\times\{1,2,\cdots,k\} \to S^1$ is the map defined by $\Phi(z,i) = \phi_{n_i}(z)$ for $z \in S^1$ and $1 \le i \le k$. Thus, $X(n_1, n_2, \cdots, n_k)$ is the union of the spaces $X(n_1)$, $X(n_2)$, $\cdots$, $X(n_k)$ with all of their natural S^1 subsets identified.

We now state our main results.

Theorem 1. If X is a compact metric space, and $f, g : X \to S^1$ are homotopic maps, then Swl(f) is homeomorphic to Swl(g).

Theorem 2. If X is a compact metric space, n is a non-zero integer, and $f : X\times S^1 \to S^1$ is the map $f(x,z) = z^n$, then Swl(f) is homeomorphic to Cyl(f).

Corollary 1. If X is a compact metric space, n is a non-zero integer, f, g : $X \times S^1 \to S^1$ are maps such that f is homotopic to g and $g(x,z) = z^n$, then Swl(f) is homeomorphic to Cyl(g).

Corollary 2. If $f : S^1 \to S^1$ is a degree $n \neq 0$ map, then Swl(f) is homeomorphic to Cyl($z \mapsto z^n$). In particular, Swl(f) is an annulus if $n = \pm 1$, and Swl(f) is a Mobius strip if $n = \pm 2$.

Theorem 3. Suppose $C_1, C_2, \cdots, C_k$ are disjoint 1-spheres in $(\partial B^3) \times S^1$, and $M^4 = (B^3 \times S^1) \cup (H_1 \cup H_2 \cup \cdots \cup H_k)$ where H_i is a 2-handle attached to $B^3 \times S^1$ along C_i, for $1 \leq i \leq k$. Let $\pi : \partial B^3 \times S^1 \to S^1$ denote the projection map, and let n_i denote the degree of the map $\pi | C_i : C_i \to S^1$ for $1 \leq i \leq k$. Then M^4 has a pseudo-spine homeomorphic to $X(n_1, n_2, \cdots, n_k)$.

Corollary 3. Suppose C is a 1-sphere in $(\partial B^3) \times S^1$, and $M^4 = (B^3 \times S^1) \cup H$ where H is a 2-handle attached to $B^3 \times S^1$ along C. Let $\pi : B^3 \times S^1 \to S^1$ denote the projection map, and suppose that the map $\pi | C : C \to S^1$ is degree one. Then M^4 has an arc pseudo-spine.

Observe that Corollary 3 includes the fact that the Mazur 4-manifold has an arc pseudo-spine.

3. Sketches of Proofs of Theorems

Proof of Theorem 1. Let f, g : $X \to S^1$ be homotopic maps.

Step 1. DblSwl(f) is homeomorphic to DblSwl(g).

Observe that DblSwl(f) = $(\cup_{x \in X} \mathfrak{F}(x)) \cup (\{-\infty,\infty\} \times S^1)$ where for each $x \in X$, $\mathfrak{F}(x)$ is the "fiber" $\{ (tx, e^{2\pi i t f(x)}) : t \in (-\infty,\infty) \}$ of DblSwl(f) which lies in $((-\infty,\infty)x) \times S^1 \subset (\Sigma(X)) \times S^1$. Similarly, DblSwl(g) = $(\cup_{x \in X} \mathfrak{G}(x)) \cup (\{-\infty,\infty\} \times S^1)$ where for each $x \in X$, $\mathfrak{G}(x)$ is the "fiber" $\{ (tx, e^{2\pi i t g(x)}) : t \in (-\infty,\infty) \}$ of DblSwl(g) which lies in $((-\infty,\infty)x) \times S^1$. For $x \in X$, in $((-\infty,\infty)x) \times S^1 \subset (\Sigma(X)) \times S^1$, we regard $(-\infty,\infty)x$ as the "vertical" direction and S^1 as the "horizontal" direction. We will describe a homeomorphism Φ of $(\Sigma(X)) \times S^1$ which carries DblSwl(f) to DblSwl(g). Φ restricts to the identity on $\{-\infty,\infty\} \times S^1$. For each $x \in X$, Φ carries $((-\infty,\infty)x) \times S^1$ onto itself; and within $((-\infty,\infty)x) \times S^1$, Φ is a "vertical" shift that moves $\mathfrak{F}(x)$ onto $\mathfrak{G}(x)$. (See Figure 3.) Explicitly, there is a vertical shift function $\sigma : X \to (-\infty,\infty)$ such that $\Phi(tx,z) = ((t+\sigma(x))x,z)$ for $(t,x) \in (-\infty,\infty) \times X$ and $z \in S^1$. σ is essentially determined by lifting the homotopy joining f to g in S^1 to a homotopy in $(-\infty,\infty)$.

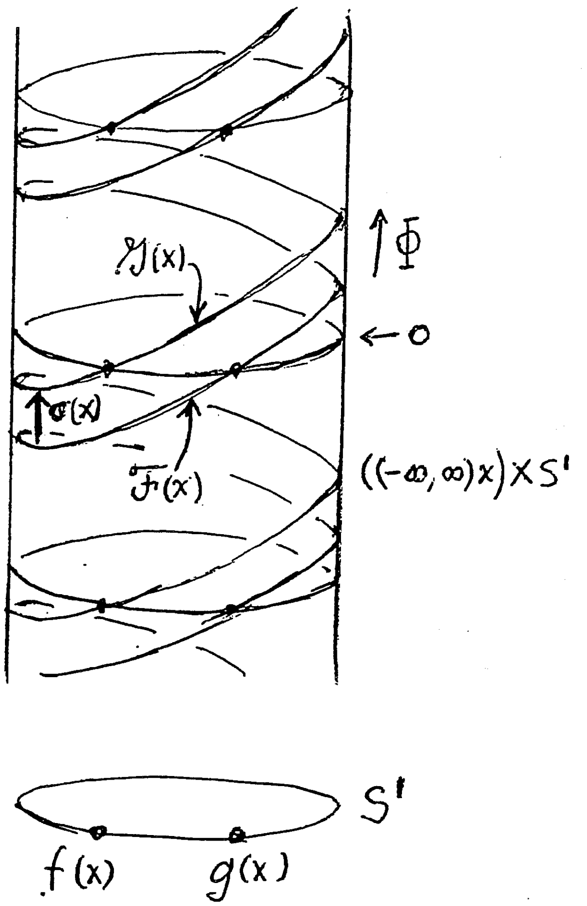


Figure 3

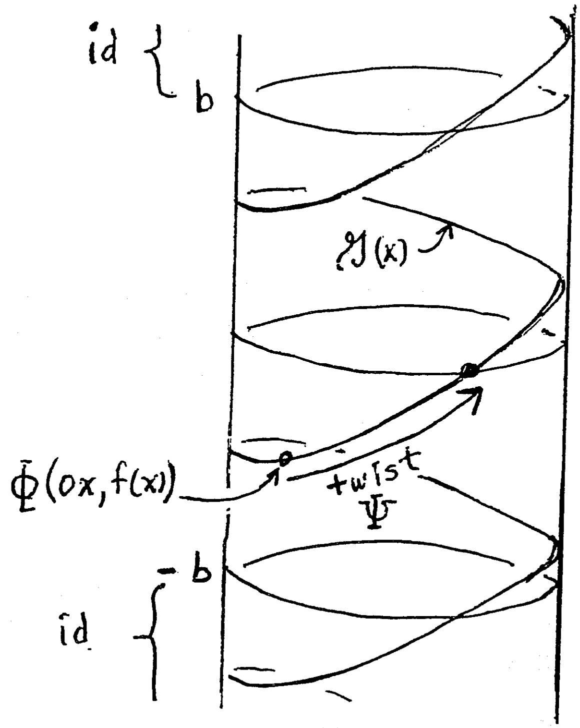


Figure 4

In more detail: suppose $h : X \times [0,1] \to S^1$ is a homotopy such that $h(x,0) = g(x)$ and $h(x,1) = f(x)$. We exploit the fact that S^1 is a group under complex multiplication to define the map $k : X \times [0,1] \to S^1$ by $k(x,t) = h(x,t)/h(x,0)$. Thus, $k(x,0) = 1$ and $k(x,1)g(x) = f(x)$ for $x \in X$. Let $e : (-\infty,\infty) \to S^1$ denote the exponential covering map $e(t) = e^{2\pi i t}$. Let $\tilde{k} : X \times [0,1] \to (-\infty,\infty)$ be the lift of k (i.e., $e \circ \tilde{k} = k$) such that $\tilde{k}(x,0) = 0$ for all $x \in X$. Define $\sigma : X \to (-\infty,\infty)$ by $\sigma(x) = \tilde{k}(x,1)$. Observe that for each $x \in X$, $f(x)/e^{2\pi i \sigma(x)} = f(x)/e(\tilde{k}(x,1)) = f(x)/k(x,1) = g(x)$. It is now straightforward to verify that a homeomorphism Φ of $(\Sigma(X)) \times S^1$ is defined by setting $\Phi(tx,z) = ((t+\sigma(x))x,z)$ for $(t,x) \in (-\infty,\infty) \times X$ and $z \in S^1$, and by requiring that $\Phi|\{-\infty,\infty\} \times S^1 = \mathrm{id}$. To verify that $\Phi(\mathrm{DblSwl}(f)) = \mathrm{DblSwl}(g)$, one shows that $\Phi(\mathfrak{F}(x)) = \mathfrak{G}(x)$ for each $x \in X$. To this end, consider a typical point $(tx, e^{2\pi i t} f(x))$ lying in the fiber $\mathfrak{F}(x)$ of $\mathrm{DblSwl}(f)$. Φ moves this point to the point $((t + \sigma(x))x, e^{2\pi i t} f(x)) = ((t + \sigma(x))x, e^{2\pi i(t + \sigma(x))} f(x)/e^{2\pi i \sigma(x)}) = ((t + \sigma(x))x, e^{2\pi i(t + \sigma(x))} g(x))$ which is a point of the fiber $\mathfrak{G}(x)$ of $\mathrm{DblSwl}(g)$.

Step 2. Swl(f) is homeomorphic to Swl(g).

Observe that $\mathrm{Swl}(f) = (\bigcup_{x\in X}\mathfrak{F}^+(x)) \cup (\{\infty\}\times S^1)$ where for each $x \in X$, $\mathfrak{F}^+(x)$ is the "fiber" $\{ (tx,e^{2\pi it}f(x)) : t \in [0,\infty) \}$ of Swl(f) which lies in $([0,\infty)x)\times S^1 \subset (C(X))\times S^1$. Similarly, $\mathrm{Swl}(g) = (\bigcup_{x\in X}\mathfrak{G}^+(x)) \cup (\{\infty\}\times S^1)$ where for each $x \in X$, $\mathfrak{G}^+(x)$ is the "fiber" $\{ (tx,e^{2\pi it}g(x)) : t \in [0,\infty) \}$ of Swl(g) which lies in $([0,\infty)x)\times S^1$. For $x \in X$, since $\mathfrak{F}^+(x) \subset \mathfrak{F}(x)$, then $\Phi(\mathfrak{F}^+(x)) \subset \Phi(\mathfrak{F}(x)) \subset \mathfrak{G}(x)$; also $\mathfrak{G}^+(x) \subset \mathfrak{G}(x)$. We will describe a homeomorphism Ψ of $(\Sigma(X))\times S^1$ which carries $\Phi(\mathrm{Swl}(f))$ onto Swl(g). Ψ combines a vertical shift with a horizontal twist in a corkscrew motion which, for each $x \in X$, carries $\mathfrak{G}(x)$ onto itself and moves $\Phi(\mathfrak{F}^+(x))$ onto $\mathfrak{G}^+(x)$. Also Ψ restricts to the identity on a neighborhood of $\{-\infty,\infty\}\times S^1$. (See Figure 4.)

Since X is compact, there is a $b \in (0,\infty)$ such that $\sigma(X) \subset (-b,b)$. It is easy to give a formula for a map $\tau : [-\infty,\infty]\times X \to [-\infty,\infty]$ such that for each $x \in X$, $t \mapsto \tau(t,x) : [-\infty,\infty] \to [-\infty,\infty]$ is an order preserving homeomorphism which restricts to the identity on $[-\infty,-b]\cup[b,\infty]$ and which carries $\sigma(x)$ to 0. (Thus, $\tau(\sigma(x),x) = 0$, and $\tau(t,x) = t$ if $|t| \geq b$.) Now define a homeomorphism Ψ of $(\Sigma(X))\times S^1$ by $\Psi(tx,z) = (\tau(t,x)x, e^{2\pi i(\tau(t,x)-t)}z)$ for $(tx,z) \in (\Sigma(X))\times S^1$. Clearly, $\Psi(tx,z) = (tx,z)$ for $|t| \geq b$; so Ψ restricts to the identity on $\{-\infty,\infty\}\times S^1$. For $x \in X$, one easily computes that Ψ moves the typical point $(tx,e^{2\pi it}g(x))$ of $\mathfrak{G}(x)$ to the point $(\tau(t,x)x, e^{2\pi i\tau(t,x)}g(x))$ which also belongs to $\mathfrak{G}(x)$; so $\Psi(\mathfrak{G}(x)) = \mathfrak{G}(x)$. Finally, for $x \in X$, $\Psi(\Phi(0x,f(x))) = \Psi(\sigma(x)x,f(x)) = (0x,e^{2\pi i(0-\sigma(x))}f(x)) = (0x,f(x)/e^{2\pi i\sigma(x)}) = (0x,g(x))$. Thus, $\Psi\circ\Phi$ is an order preserving homeomorphism from $\mathfrak{F}(x)$ onto $\mathfrak{G}(x)$ which carries the boundary point of $\mathfrak{F}^+(x)$ to the boundary point of $\mathfrak{G}^+(x)$. Consequently, $\Psi\circ\Phi(\mathfrak{F}^+(x)) = \mathfrak{G}^+(x)$. It follows that $\Psi\circ\Phi(\mathrm{Swl}(f)) = \mathrm{Swl}(g)$. □

Proof of Theorem 2. $f : X\times S^1 \to S^1$ satisfies $f(x,z) = z^n$ where $n \neq 0$. Both Cyl(f) and Swl(f) can be regarded as subsets of $C(X\times S^1)\times S^1$. We will describe a homeomorphism Φ of $C(X\times S^1)\times S^1$ which carries Cyl(f) onto Swl(f). Φ achieves this result by twisting in the S^1-direction in the $C(X\times S^1)$ factor of $C(X\times S^1)\times S^1$. For $t(x,z) \in C(X\times S^1)$ and $w \in S^1$, set $\Phi(t(x,z),w) = (t(x,e^{-2\pi it/n}z),w)$ and $\Phi(\infty,w) = (\infty,w)$. This clearly defines a homeomorphism of $C(X\times S^1)\times S^1$. Let $(x,z) \in X\times S^1$ and consider a typical point $(t(x,z),f(x,z)) = (t(x,z),z^n)$ of the fiber emanating from (x,z) in Cyl(f). Set $z' = e^{-2\pi it/n}z$. Then $\Phi(t(x,z),f(x,z)) = (t(x,e^{-2\pi it/n}z),z^n) = (t(x,e^{-2\pi it/n}z),e^{2\pi it}(e^{-2\pi it/n}z)^n) = (t(x,e^{-2\pi it/n}z),e^{2\pi it}f(x,e^{-2\pi it/n}z)) = (t(x,z'),e^{2\pi it}f(x,z'))$ which is a typical point of the fiber $\mathfrak{F}^+(x,z')$ in Swl(f). It follows that $\Phi(\mathrm{Cyl}(f)) = \mathrm{Swl}(f)$. □

Corollaries 1 and 2 are obvious consequences of Theorems 1 and 2.

Proof of Theorem 3. Recall that $\pi : \partial B^3 \times S^1 \to S^1$ denotes the projection map. Clearly $\mathrm{Cyl}(\pi)$ is homeomorphic to $B^3 \times S^1$. On the other hand, since $\pi(x,z) = z$, then by Theorem 2, $\mathrm{Cyl}(\pi)$ is homeomorphic to $\mathrm{Swl}(\pi)$. Hence, we identify $B^3 \times S^1$ with $\mathrm{Swl}(\pi)$.

Recall that $C_1, C_2, \cdots, C_k$ are disjoint 1-spheres in $(\partial B^3) \times S^1$, and $M^4 = (B^3 \times S^1) \cup (H_1 \cup H_2 \cup \cdots \cup H_k)$ where H_i is a 2-handle attached to $B^3 \times S^1$ along C_i, for $1 \leq i \leq k$. Think of the 2-handle H_i as the product of two 2-dimensional disks D_i and E_i with $d_i \in \mathrm{int}(D_i)$ and $e_i \in \mathrm{int}(E_i)$ so that $H_i \cap (B^3 \times S^1) = (\partial D_i) \times E_i$ and $C_i = (\partial D_i) \times \{e_i\}$. (Thus, $D_i \times \{e_i\}$ is the "core" and $\{d_i\} \times E_i$ is the "cocore" of H_i.) Recall that $n_i = \deg(\pi | C_i)$. Set $X = \bigcup_{i=1}^{k} \mathrm{Swl}(\pi | C_i) \cup (D_i \times \{e_i\})$. Then clearly X is homeomorphic to $X(n_1, n_2, \cdots, n_k)$. (M^4 and X are represented schematically in Figure 5.)

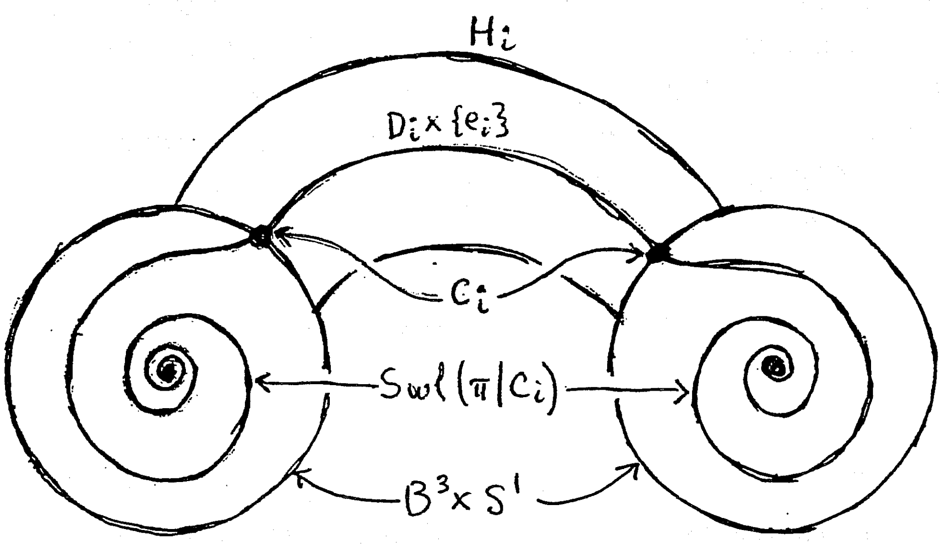


Figure 5

It remains to describe a homeomorphism $h : (\partial M^4) \times [0,\infty) \to M^4 - X$. Note that $(B^3 \times S^1) - X = \mathrm{Swl}(\pi) - X$ is the union of the fibers of $\mathrm{Swl}(\pi)$ that emanate from the points of $(\partial B^3 \times S^1) - \bigcup_{i=1}^{k} C_i$. Each of these fibers is homeomorphic to $[0,\infty)$. We will "enlarge" this fibering to a fibering of all of $M^4 - X$ by copies of $[0,\infty)$. Let $x \in \partial M^4$. If $x \in \partial B^3 \times S^1$, then $h(\{x\} \times [0,\infty))$ is the fiber of $\mathrm{Swl}(\pi)$ emanating from x. If $x \notin \partial B^3 \times S^1$, then $x \in (\mathrm{int}(D_i)) \times (\partial E_i)$ for some i, $1 \leq i \leq k$. If $x \in \{d_i\} \times (\partial E_i)$, then $h(\{x\} \times [0,\infty))$ is the radius of the disk $\{d_i\} \times E_i$ joining the center point (d_i,e_i) to x, minus the center point (d_i,e_i). If $x \in$

$(\mathrm{int}(D_i) - \{d_i\}) \times (\partial E_i)$, then $h(\{x\} \times [0,\infty))$ is the union of a curved arc joining x to a point $y \in (\partial D_i) \times (\mathrm{int}(E_i) - \{e_i\})$ together with the fiber of $\mathrm{Swl}(\pi)$ emanating from y. (See Figure 6.)

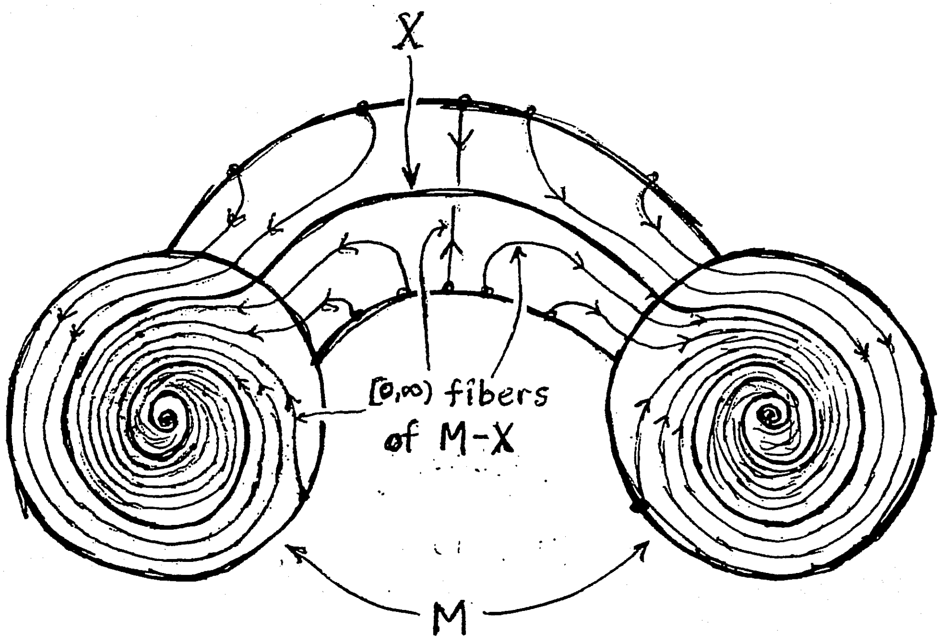


Figure 6

Proof of Corollary 3. Theorem 3 implies that M^4 has a pseudo-spine homeomorphic to $X(1)$. Clearly $X(1)$ is a disk. According to [3], $X(1)$ can be "squeezed" to an arc in $\mathrm{int}(M^4)$. (Interpreted literally, [3] applies only in manifolds of dimension 3. However, the methods of [3] work in manifolds of all dimensions ≥ 3. This is fully explained on page 95 of [2].) Thus, M^4 has an arc pseudo-spine. □

4. Conjectures

The techniques and results about pseudo-spines of 4-manifolds presented here are rather modest and restricted. Although at present we have no idea how to enlarge the scope of these results, we are undaunted in formulating conjectures of much greater breadth and boldness.

Conjecture 1. If a compact 4-manifold with is homotopy equivalent to $X(n_1, n_2, \cdots, n_k)$, then it has a pseudo-spine homeomorphic to $X(n_1, n_2, \cdots, n_k)$.

Conjecture 2. Every homotopy 4-ball has an arc pseudo-spine.

We break Conjecture 2 into two conjectures.

Conjecture 2A. Every PL homotopy 4-ball has a handlebody decomposition with no 3- or 4-handles.

Conjecture 2B. If a homotopy 4-ball has a handlebody decomposition with no 3- or 4-handles, then it has an arc pseudo-spine.

Conjecture 3. If two compact 4-manifolds have arc pseudo-spines, then so does their boundary-connected sum.

Conjecture 4. If a compact 4-manifold has a tree pseudo-spine, then it has an arc pseudo-spine.

Conjecture 5. If a compact 4-manifold has a pseudo-spine which is homeomorphic to a 1-dimensional polyhedron, then is has a pseudo-spine which is homeomorphic to a wedge of circles.

Conjecture 6. The 4-ball is the only homotopy 4-ball that has disjoint pseudo-spines.

In connection with Conjecture 6, we note that in [4] it is shown that for $n \geq 9$, there are homotopy n-balls distinct from the n-ball that have disjoint spines. Conjecture 6 asserts that the situation is different in dimension 4.

References

1. F. D. Ancel and C. R. Guilbault, Compact contractible n-manifolds have arc spines ($n \geq 5$), to appear in *Pacific J. Math.*
2. R. J. Daverman, *Decompositions of manifolds,* Academic Press, Orlando, Florida, 1986.
3. R. J. Daverman and W. T. Eaton, An equivalence for the embeddings of cells in a 3-manifold, *Trans. Amer. Math. Soc.* 145 (1969), 369-381.
4. C. R. Guilbault, Some compact contractible manifolds containing disjoint spines, to appear in *Topology.*
5. C. H. Giffen, Disciplining dunce hats in 4-manifolds, unpublished manuscript, 1976.
6. B. Mazur, A note on some contractible 4-manifolds, *Annals of Math.* 73 (1961), 221-228.

Department of Mathematical Sciences
University of Wisconsin - Milwaukee
Milwaukee, WI 53201

ancel@csd4.csd.uwm.edu

craigg@csd4.csd.uwm.edu

MOST MANIFOLDS ARE PL FIBRATORS

by

Robert J. Daverman

The title is intended to express a philosophy and isn't meant to convey precise mathematical content. The philosophy emerged in [D3], where the results suggest that manifolds in a fairly extensive collection act as PL fibrators. Among closed, orientable 2-manifolds the exceptions are known to consist only of the 2-sphere and the torus. Among closed, orientable 3-manifolds, those having either hyperbolic, Sol, or $\widetilde{SL_2(R)}$ geometric structure were shown to be PL fibrators, as were basically all those with infinite first homology arising as connected sums of at least two nonsimply connected manifolds. The chief result establishes that an aspherical, virtually geometric 3-manifold is a PL fibrator if it is one in codimension 2. The investigation uncovered methodology for treating a related matter involving limited codimension: any closed, orientable manifold with (k-1)-connected compact universal cover is a codimension k PL fibrator if it is one in codimension 2.

To explain, here is some notation and fundamental terminology: M is a connected, orientable, PL (n+k)-manifold, B is a polyhedron, and $p:M \to B$ is a PL map such that each $p^{-1}b$ has the homotopy type of a closed, connected n-manifold. For a fixed orientable n-manifold N, such a PL map $p:M \to B$ is said to be N-*like* if each $p^{-1}b$ collapses to an n-complex homotopy equivalent to N. Then N is a *codimension* k *PL fibrator* if, for every orientable (n+k)-manifold M and N-like PL map $p:M \to B$, p is an approximate fibration; when it has this property for all $k>0$, N is a *PL fibrator*. Worth emphasizing, or admitting, is the fact that the PL tameness feature required of an N-like map, the collapsibility aspect, imposes significant homotopy-theoretic relationships between N and preimages of links in B.

Remarkably, at this stage of development only two types of manifolds are known not to be PL fibrators – those that already fail in codimension 2 and those that have a sphere as Cartesian factor. The codimension 2 situation is fairly well understood (cf. the Introduction to [D2]). It is a good locale for understanding non-fibrators. The classic example is S^1, seen in any Seifert fibration $p:S^1\times B^2 \to B^2$ with irregular (circle) fibers. Any manifold N which regularly, cyclically covers itself fails to be a codimension 2 fibrator by more or less the same construction, and the 3-manifolds with Nil Geometric structure that are circle bundles over the torus fail to be codimension 2 fibrators but do not admit regular, cyclic self-coverings [D1]. The n-sphere fails to be a codimension n+1 fibrator, as seen from a map $S^n\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ having as preimages $S^n\times 0$ and all spheres of the form $z\times(r\cdot S^n)$, $r>0$; this can be adjusted to a PL S^n-like map.

Previous work describing PL fibrators typically required the fundamental groups to contain no nontrivial, Abelian normal subgroups. One nice class of such groups consists of nontrivial free products. Not surprisingly, then, there are fairly strong results yielding that nontrivial connected sums are PL fibrators. Additional techniques became imperative for more extensive analysis of 3-manifolds, due to the desire to understand certain Seifert fiber spaces, in which the fundamental groups contain infinite cyclic normal subgroups.

Here are some other items of terminology. A proper map $p:M \to B$ between locally compact ANR's is called an *approximate fibration* if it satisfies a standard approximate homotopy lifting property: given an open cover Ω of B, an arbitrary space X, and two maps $f:X \to M$ and $F:X\times I \to B$ such that $pf = F_0$, there exists a map $F':X\times I \to M$ such that $F'_0 = f$ and pF' is Ω-close to F (in the sense that to each $z\in X\times I$ there corresponds $U_z\in\Omega$ with $\{F(z),pF'(z)\} \subset U_z$).

A group Γ is *hopfian* if every epimorphism $\Psi:\Gamma \to \Gamma$ is an automorphism, while it is *cohopfian* if every monomorphism $\Phi:\Gamma \to \Gamma$ is an automorphism. Two related concepts useful for sorting out fibrator properties are: Γ is *normally cohopfian* if every monomorphism $\Phi:\Gamma \to \Gamma$ with normal image is an automorphism, and Γ is *hyperhopfian* if every homomorphism $\psi:\Gamma \to \Gamma$ with $\psi(\Gamma)$ normal and $\Gamma/\psi(\Gamma)$ cyclic is an automorphism.

STATEMENTS OF THE MAIN RESULTS

Let $\mathfrak{R}$ denote the class of "non-bogus", closed, orientable 3-manifolds, namely, the closed orientable ones that are virtually geometric. A 3-manifold (without boundary, for our purposes) is *virtually geometric* if it is finitely covered by a geometric one, meaning that the covering space is a connected sum of 3-manifolds which are either Haken or have some geometric structure. Recall that a 3-manifold N is *irreducible* if every PL 2-sphere in N bounds a 3-cell there, and a compact N is *Haken* if it is irreducible and contains an incompressible (closed) surface.

Theorem 1. An aspherical 3-manifold N in $\mathfrak{R}$ is a PL fibrator if and only if it is a codimension 2 PL fibrator.

As a corollary, every aspherical 3-manifold N in $\mathfrak{R}$ which is neither a Seifert fiber space nor a surface bundle over S^1 is a PL fibrator, as its fundamental group must be hyperhopfian, indicating N is a codimension 2 fibrator [D2]. The same was already known for 3-manifolds in $\mathfrak{R}$ expressed as a connected sum of at least two nonsimply connected, irreducible 3-manifolds, one of which has infinite fundamental group.

Theorem 2. Suppose N^n has a closed (k-1)-connected univeral cover (k>2) . Then N^n is a codimension k PL fibrator if and only if it is a codimension 2 fibrator.

TECHNIQUES

The following gives a basic method for detecting fibrators.

Lemma 1. If a closed, aspherical n-manifold N^n is a codimension 2 fibrator, and if $\pi_1(N^n)$ is normally cohopfian and has no Abelian normal subgroup $A \neq 1$ with $\pi_1(N^n)/A$ isomorphic to a normal subgroup of $\pi_1(N^n)$, then N^n is a PL fibrator.

A key step involves showing that an aspherical 3-manifold which is a codimension 2 PL fibrator, unless it has Nil geometric structure, satisfies all the other hypotheses of Lemma 1. Most of the effort to obtain the normally cohopfian property was provided by González-Acuña and Whitten [GW], who improved upon an already strong result due to Wang and Wu [WW].

Lemma 2 [GW]. If N^3 is a closed, virtually geometric 3-manifold such that $\pi_1(N^3)$ is not cohopfian, then N^3 is a Seifert fiber space with one of the following geometric structures: E^3, $S^2\times\mathbb{R}$, $H^2\times\mathbb{R}$, or Nil.

Nil 3-manifolds contravene the pattern set by other Seifert fibered spaces: they can be codimension 2 PL fibrators without possessing normally cohopfian fundamental groups. The circle bundle over the Klein bottle with fundamental group

$$G = \langle a,b,k \mid a^{-1}ka=k^{-1}=b^{-1}kb,\ k^2=a^2b^2\rangle$$

is a prime example, for one can check that a, b^2, k^2 generate a normal subgroup of G isomorphic to G, and therefore the associated covering space is another copy of the source manifold. In identical manner circle bundles over the torus regularly cover themselves, but they are less interesting, since they never serve as codimension 2 fibrators [D1].

The quotient of G by the normal subgroup in the example above is $Z_2\oplus Z_2$. Direct sums of this type arise as subgroups of the deck transformations associated with coverings stemming from the failure of normal cohopficity and ultimately play an instrumental role in the main result.

Lemma 3. If the closed 3-manifold N^3 has Nil geometric structure, and if $\Gamma\neq 1$ is a group acting freely on N^3 such that N^3/Γ is homotopy equivalent to N^3, then Γ contains a subgroup of the form $Z_d\oplus Z_d$ and hence acts freely on no homology sphere.

See [Br, p 181] for the Smith theory about nonexistence of such actions on homology spheres. The rest of the statement depends upon uniqueness of the Seifert data for N^3 [S2].

With Nil manifolds Lemma 3 supplants the normal cohopficity feature prevalent in most aspherical 3-manifolds for the proof of Theorem 1, and with Haken 3-manifolds Lemma 5 supplants the non-existence of Abelian normal subgroups feature which has proved so useful in other settings. Before getting to the latter we mention a result complementary to Lemma 3.

Lemma 4. If N^n is a closed n-manifold and Γ is a finite group that acts (PL) freely, preserving orientations, on both S^k and on N^n such that the orbit space N^n/Γ is homotopy equivalent to N^n, then N^n fails to be a codimension k+1 (PL) fibrator.

The orbit space construction of [D3, Example 2B] shows how to produce the required N^n-like map.

Lemma 5. No k-manifold T, $k\geq 3$, satisfies all of the following homotopy data: $\pi_1(T)$ is infinite; $\pi_2(T)$ is free Abelian of rank r, $1<r<\infty$; and $\pi_i(T) = 0$ for $2<i<k$; moreover, there is no such T in the $r=1$ case unless $\pi_1(T)$ contains an infinite cyclic subgroup of finite index.

With even k, this follows by comparing the homology data of T', the universal cover of T, with that of $K(\pi_2(T),2)$, naturally computed in the Cartesian product of r copies of CP^∞; with odd k, one must also invoke results of Epstein about ends [Ep].

Finally, somewhat like Lemma 5 in Theorem 1, the result below plays a major part in the proof of Theorem 2. The crucial ingredient comes from [EM].

Lemma 6. If Π is a finite Abelian group and Σ is a closed k-manifold such that $\pi_1(\Sigma)=0$, $\pi_2(\Sigma)=\Pi$, and $\pi_i(\Sigma)=0$ for $2<i<k$, then Π is trivial.

Question. Are there exceptions besides the 3-sphere to the statement: a closed 3-manifold N^3 is a PL fibrator if and only if it is a codimension 2 fibrator? Within the class $\mathfrak{R}$ this is open for 3-manifolds covered by S^3 and for the ones arising as connected sums of manifolds with nontrivial, finite fundamental groups (other than exactly two summands, both with $\pi_1 = Z_2$).

REFERENCES

[Br] G. E. Bredon, INTRODUCTION TO COMPACT TRANSFORMATION GROUPS, Academic Press, New York, 1972.

[D1] R. J. Daverman, *3-Manifolds with geometric structures and approximate fibrations,* Indiana Univ. Math. J. **40** (1991), 1451-1469.

[D2] __________, *Hyperhopfian groups and approximate fibrations,* Compositio Math. **86** (1993), 159-176.

[D3] __________, *Manifolds that induce approximate fibrations in the PL category,* submitted.

[EM] S. Eilenberg and S. MacLane, *On the groups* H(Π,n), *II,* Annals of Math. (2) **70** (1954), 49-139.

[Ep] D. B. A. Epstein, *Ends,* in TOPOLOGY OF 3-MANIFOLDS AND RELATED TOPICS (M. K. Fort, ed.), Prentice Hall, Englewood Cliffs, N. J. (1962), pp. 110-117.

[GW] F. González-Acuña and Wilbur Whitten, *Cohopficity of three-manifold groups,* Topology Appl., to appear.

[He] J. Hempel, 3-MANIFOLDS, Annals of Math. Studies #86, Princeton Univ. Press, Princeton, N. J., 1976.

[S1] P. Scott, *The geometries of 3-manifolds,* Bull. London Math. Soc. **15** (1983), 401-487.

[S2] ________, *There are no fake Seifert fiber spaces with infinite* π_1, Annals of Math. (2) **117** (1983), 35-70.

[WW] S. Wang and Y.-Q. Wu, *A covering invariant of graph manifolds,* preprint.

Department of Mathematics
The University of Tennessee
Knoxville, TN 37996-1300

Disks with Bad Boundaries

Paul Fabel

We investigate the topology of X, the space of self-homeomorphisms of the 2-sphere which pointwise fix a non-separating continuum K.

We pose strategies for building a contraction of X that approximates X by subspaces, which are known to be absolute retracts by a theorem of Mason. If successful, we can then conclude by a theorem of Hanner that X itself is an AR. It will then follow from theorems of D.W. Henderson, and of Torunczyk and Dobrowolski that X is homeomorphic to separable Hilbert space.

We remark on two important facets of our construction.

Given a closed disk D in the plane and distinct points a,b on the boundary of D, we consider the the subset F of D consisting of those points x in D which are not separated in D, from {a,b}, by any chord of D. We assert that F is a closed arc, that F is continuous in the data, and that whenever it's meaningful, F is as short as possible.

Each closed disk in the plane can be canonically parameterized by a conformal map of the unit disk.

Version 1.1 Aug 93

Negatively Curved Groups have the Convergence Property

Eric M. Freden *

ABSTRACT. It is known that the Cayley graph Γ of a negatively curved (Gromov-hyperbolic) group G has a well-defined boundary at infinity $\partial\Gamma$. Furthermore, $\partial\Gamma$ is compact and metrizable. In this paper I show that G acts on $\partial\Gamma$ as a convergence group. This implies that if $\partial\Gamma \simeq \mathbf{S}^1$, then G is topologically conjugate to a cocompact Fuchsian group.

0. Introduction

The theory of convergence groups was first introduced by Gehring and Martin [GM] as a natural generalization of Möbius groups. All discrete quasiconformal groups display the convergence property and at first glance this fact would seem to imply a much larger class of groups. However, the work of Tukia [T], Gabai [Ga], et al, shows that every convergence group acting on S^1 is in fact conjugate to a Möbius group by a homeomorphism of S^1 (compare with 3.3 of [Gr]). Although this is false in the case of S^2, all known counterexamples share the same construction technique [MS].

This paper deals exclusively with *discrete* (in the compact-open topology) convergence groups acting on metric spaces. Let (X,d) be such a metric space. In most of the literature, X is either S^n or B^n although many results can be generalized. We say that $G \subset$Homeo(X) is a *convergence group* if given any sequence $\{g_m\}$ of distinct group elements, there exist (not necessarily distinct) points $x, y \in X$ and a subsequence $\{g_n\}$ such that

$$g_n(z) \to x \quad \text{locally uniformly on } X \setminus \{y\}, \text{ and}$$
$$g_n^{-1}(z) \to y \quad \text{locally uniformly on } X \setminus \{x\}.$$

(Here "locally uniformly" means uniformly on compact subsets, i.e. if $C \subset X \setminus \{y\}$ is compact and U is a neighborhood of x, then $g_n(C) \subset U$ for all sufficiently large n.) Tukia has termed the above criteria (CON) and I will do the same.

The idea of negatively curved groups is due to Gromov [Gr]. Other synonyms in current use are *Gromov hyperbolic, word hyperbolic* or merely *hyperbolic*. The fact that Gromov came up with a good idea has become evident in the last few years—the entire research area of geometric group theory has exploded with the introduction of negative curvature. All negatively curved groups are finitely presented. A nice argument showing that negative curvature is a group invariant can be found in [Sw]. The word and conjugacy problems can always be solved for negatively curved groups [C1] [Gr]. More recently Sela [S] has announced that the isomorphism problem is also solvable in negatively curved groups. Since negatively curved groups are in some sense generic [O], the unsolvability of the above problems is the exception rather than the rule for finitely presented groups.

* This research was supported in part by the Geometry Center, University of Minnesota, an STC funded by the NSF, DOE, and Minnesota Technology, Inc; and by an NSF research grant; also by Brigham Young University.

This paper links negatively curved groups and convergence groups. Some results in this area are already known. For instance, let M denote a closed riemannian n-manifold with all sectional curvatures less than some negative constant. By Toponogov's Comparison Theorem [CE], the universal cover $\widetilde{M}$ has thin triangles. The group $G = \pi_1(M)$ acts by isometry on both its Cayley graph Γ and $\widetilde{M}$; hence they are quasi-isometric [C2] [GH]. Gromov has shown that quasi-isometry is a group invariant [Gr], therefore G is negatively curved. Martin and Skora [MS] show that G also acts as a convergence group on the boundary at infinity of $\widetilde{M}$. The main theorem in this paper generalizes the above via combinatorial methods.

In section 1, I review definitions and some known results. Section 2 contains some technical lemmas. The main theorem is proved in section 3 along with some related material. The concluding pages provide drawings to accompany almost every combinatorial result. (I find re-reading my own theorems almost impossible without the figures.) I acknowledge with gratitude the encouragement of Jim Cannon. Eric Swenson developed some of the techniques and groundwork used in section 1. Greg Conner, Dave Gabai, Steve Humphries, Gaven Martin, and Bernard Maskit provided useful comments. I am indebted to Brigham Young University for a stimulating research environment as well as for travel support.

1. Preliminaries

Let (X,d) be a metric space, with $a, b \in X$. A *path* connecting a and b is the image of a continuous function $\alpha : [0,1] \to X$ satisfying $\alpha(0) = a$ and $\alpha(1) = b$. The *length* of this path is defined by

$$length(\alpha) = \sup \sum_{i=1}^{n} \mathrm{d}(\alpha(x_{i-1}), \alpha(x_i))$$

where the supremum is taken over all finite partitions $\{0 = x_0, x_1, \cdots x_n = 1\}$ of $[0,1]$. X is a *path metric* space if for any $a, b \in X$ there exists a path α connecting a and b with $\mathrm{d}(a,b) = length(\alpha)$. Any such path realizing the distance between endpoints is referred to as *geodesic*. If there is no ambiguity it is easier to denote a geodesic path between a and b by $\overline{ab}$. A *ray* is the image of a continuous map $R : [0, \infty) \to X$. (It is convenient to refer to both the map and its image as R, and to refer to $R(t)$ merely as t if the context is clear.) The ray R is geodesic if every finite sub-segment of R is geodesic, i.e. R is an isometry onto its image. Geodesic lines are defined similarly.

Let (X,d) be a path metric space. A (geodesic) *triangle* $\Delta(a, b, c) \subset X$ consists of three distinct points $a, b, c \in X$ (called *vertices*) and three geodesic segments $\overline{ab}$, $\overline{bc}$, and $\overline{ac}$ (called *edges*). It is not true in general that such a triangle is determined by its vertices (consider $X = S^2$).

Definition 1.1: Let $\delta \geq 0$. A path metric space (X,d) is *negatively curved* (δ) or δ *-hyperbolic* if for each geodesic triangle $\Delta(a, b, c) \subset X$ and for each $x \in \overline{ab}$ it is true that

$$\mathrm{d}(x, \overline{bc} \cup \overline{ac}) < \delta.$$

The triangle $\Delta(a,b,c)$ is said to be *δ-thin.*

In most of the literature the above inequality is not strict—however, several of the lemmas proved below are shorter using strict inequality. It is easy to see that the two definitions are equivalent.

The thin triangles condition has been attributed to Rips. There are many equivalent definitions of negative curvature, for example, exponential divergence of rays [C2], an inequality with respect to the "overlap" or generalized inner-product [Gr], and a linear inequality relating area to perimeter [Gr]. Thin triangles seems to be the most intuitive of the above and will be used in this paper. Let $\epsilon > 0$. Examples of negatively curved path metric spaces are trees which have ϵ-thin triangles and H^n, hyperbolic n-space, which has thin triangles with $\delta = \log(1+\sqrt{2}) + \epsilon$.

Exercise 1.2: If X has thin triangles (δ), then quadrilaterals are 2δ-thin.

Definition 1.3: Let G denote a group with finite generating set C, closed with respect to inverses. Let $\Gamma = \Gamma(G,C)$ denote the *Cayley graph* of G with respect to the given generating set. This is a simplicial 1-complex with one vertex for each group element. The directed edge set is $E = \{(h,c,hc) : h \in G, c \in C\}$ where h, hc represent the initial and terminal vertices respectively, and c is the label of the segment in between.

It is not hard to see that Γ is homogeneous—every vertex looks like every other vertex. It is often convenient to pick a specific vertex as an origin. Usually this vertex will be denoted as 0 and will correspond to the group identity. The group G acts on Γ by left multiplication. If h is a vertex, (h,c,hc) is a directed edge and $g \in G$, then $g \cdot h = gh$ represents another vertex and $g \cdot (h,c,hc)$ is the directed edge (gh, c, ghc). Depending on the context, a word $c_1c_2....c_k$ can represent

i) a group element in G

ii) the vertex of Γ labelled $c_1c_2....c_k$

iii) the edge path from 0 to the vertex in *ii)*

iv) an edge path from any vertex h to the vertex $hc_1c_2....c_k$.

Consider each edge as being isometric to the unit interval. In this way Γ becomes a locally compact path metric space. The distance function is referred to as the *word metric.* A minimal representation for a group element h becomes a geodesic edge path from 0 to h; relators correspond to closed loops in Γ. By definition, G acts on Γ freely and properly discontinuously as a group of isometries. For each generator $c \in C$, consider the set of open half edges emanating from 0. The union of these edges, along with the vertex 0 forms a fundamental domain D for the group action, as can be seen by checking the following conditions:

i) $g(D) \cap D = \emptyset$ for all g except the identity

ii) Every $z \in \Gamma$ is G equivalent to a point in D

iii) The "sides" of $\overline{D}$ are paired by elements of G

iv) If $K \subset \Gamma$ is compact, $g(\overline{D}) \cap K = \emptyset$ except for finitely many $g \in G$

Evidently Γ/G is a bouquet of circles, with one circle for each generator. Any compact subset of Γ is contained in the union of finitely many edges and vertices. Bearing this in mind, the following is immediate.

Exercise 1.4: If $V, W \subset \Gamma$ are compact sets, then $g(V) \cap W = \emptyset$ for all but finitely many $g \in G$.

Call a metric space *proper* if the closure of every metric ball is compact (i.e. the Heine-Borel theorem holds).

Definition 1.5: A finitely generated group G is *negatively curved*, or *word hyperbolic* if the corresponding Cayley graph Γ is negatively curved.

Definition 1.6: Let (X,d) be a proper path metric space with δ-thin triangles for some fixed $\delta > 0$. Two geodesic rays $R, S : [0, +\infty) \to X$ are *equivalent*, written $R \sim S$, if

$$\limsup_{t \to +\infty} \mathrm{d}(R(t), S(t)) < +\infty.$$

Another way to say this is that R strays at most a bounded distance from S and vice versa. Indicate by $[R]$ the equivalence class containing R.

Exercise 1.7. In fact, the bounded distance mentioned above is (asymptotically) at most 2δ. (Hint: quadrilaterals are 2δ-thin.)

One might worry about rays that start at different places. In fact, this is usually not a problem. Given any ray R and point $x \in X$ there is a ray S starting from x that is equivalent to R (see I.2 of [Sw]). Therefore if x and y are distinct points of X, there is a bijection between the set of ray classes starting at x and those starting at y.

Definition 1.8: The boundary at infinity ∂X is defined as the set of equivalence classes of (geodesic) rays.

It is necessary to put a topology on ∂X that is independent of ray base points. The following is a generalization of classical hyperbolic geometry.

Definition 1.9: If R is a ray in X (or more generally any closed set) and $x \in X$, define a relation (multi-valued "function") by

$$\mathbf{p}_R(x) = \{r \in R : \mathrm{d}(x, r) = \mathrm{d}(x, R)\}.$$

The set $\mathbf{p}_R(x)$ is called the *closest point projection* of x into R.

Observe that if $p \in \mathbf{p}_R(x)$ and $t \in \overline{xp}$ then $p \in \mathbf{p}_R(t)$ also. In the special case $X = \Gamma$, a negatively curved Cayley graph, if x is a point and R a geodesic (segment, ray, line) there can be at most finitely many points (all necessarily vertices) in $\mathbf{p}_R(x)$. Furthermore, $\mathbf{p}_R$ is a "continuous" relation, in the sense that if x is sufficiently close to y, then $\mathbf{p}_R(x)$ is in a neighborhood of $\mathbf{p}_R(y)$.

Definition 1.10: Let R be a ray and $r \in R$. Define the *halfspace determined by R and r* as

$$H(R, r) = \{x \in X : \mathrm{d}(x, R[r, +\infty)) \le \mathrm{d}(x, R[0, r))\}.$$

Set $H^-(R, r) = X \setminus H(R, r)$. Call $H^-(R, r)$ the *complementary halfspace*.

A point is in $H^-(R,r)$ if *all* of its projections into R lie in the intial segment $R[0,r)$. If at least one of its closest projections lies in the subray $R[r,+\infty)$, then the point is in $H(R,r)$. Although a halfspace and its complement are defined differently, they are almost indistinguishable in the large. Any theorem proved about halfspaces is true (or has an analog) for complementary halfspaces. The halfspaces yield neighborhoods for ∂X in the following way.

Definition 1.11: Given a halfspace $H(R,r)$, define an (open) *disk at infinity* by

$$D(R,r) = \{[S] : S \text{ is a ray, and } \liminf_{s\to+\infty} \mathrm{d}(S(s), H^-(R,r)) = +\infty\}.$$

Let $D^-(R,r)$ signify $X \setminus D(R,r)$.

Swenson [Sw] gives a nice argument showing that the disks at infinity form a base for the same topology on ∂X used by Gromov, et al, and verifies independently that ∂X is compact, metrizable, and finite dimensional. The compactification $\overline{X} = X \cup \partial X$ can be given a global metric which induces the original topology on X and agrees with that of ∂X. I prefer, however, to ignore the global metric in favor of combinatorial arguments dealing with halfspaces and disks. This is particularly relevant in the case that X is the Cayley graph Γ of a negatively curved group.

2. Some Geometric Properties of Γ

Let G be a negatively curved group with fixed (finite) generating set C and associated Cayley graph Γ. One can define an extension of the action of G to $\partial\Gamma$ in the obvious way: if $R \subset \Gamma$ is a (geodesic) ray, set $g([R]) = [g(R)]$. Suppose that S is another ray equivalent to R. Then by definition there exists $N \in \mathbf{Z}^+$ such that for all $r \in R$ and all $s \in S$, both $\mathrm{d}(r,S) < N$ and $\mathrm{d}(s,R) < N$. But every $g \in G$ is an isometry on Γ, so $\mathrm{d}(g(r), g(S)) < N$ and $\mathrm{d}(g(s), g(R)) < N$. Therefore $g(R)$ and $g(S)$ are equivalent rays, meaning that the action on $\partial\Gamma$ is well-defined.

In light of the above, one may dispense with equivalence classes of rays and use individual representatives. Since G is a group, it follows that $g : \partial\Gamma \to \partial\Gamma$ is a bijection. To show that G acts as a group of homeomorphisms, it suffices to show that each g^{-1} is continuous on $\partial\Gamma$. Observe that g maps halfspaces to halfspaces, hence basic disk neighborhoods to basic disk neighborhoods. Thus g is an open map, so g^{-1} is continuous.

Definition 2.1 : A negatively curved group is *elementary* if $\partial\Gamma$ contains at most two points. It is *non-elementary* otherwise.

Elementary groups are either torsion or virtually cyclic depending on whether $\partial\Gamma$ is empty or contains exactly two points (see, e.g. the discussion following II.17 in [Sw]). On the other hand, any non-elementary negatively curved group G contains a rank two free subgroup, and hence $\partial\Gamma$ is in fact uncountable (see 8.2 of [Gr]). In the non-elementary case, G is a discrete subgroup of Homeo($\partial\Gamma$) using the compact-open topology. The argument is a transparent consequence of the convergence conditions (CON) which are established in the proof of theorem 3.4. The action of G on $\partial\Gamma$ is not necessarily effective (meaning nonidentity elements can act trivially). Let H be the subgroup of G that acts trivially on $\partial\Gamma$. Clearly H is normal, and (CON) shows that H must be finite. The quotient $G_0 = G/H$ acts effectively.

The next four properties are generalizations of hyperbolic geometry applied to Γ. The first result is "folklore". It says that there is a (not necessarily unique) geodesic between every two points at infinity.

Lemma 2.2. *If $R, S \subset \Gamma$ are inequivalent rays, then there is a geodesic line $P : \mathbf{R} \to \mathbf{R}$ such that $P^- = P(-\infty, 0]$ is equivalent to R and $P^+ = P[0, +\infty)$ is equivalent to S.*

Proof: We may assume that R and S both emanate from the origin of Γ. Parametrize the rays by arc length. For each $n \in \mathbf{Z}^+$ let r_n (resp. s_n) denote the point on R (resp. S) with $\mathrm{d}(0, r_n) = n$ (resp. $\mathrm{d}(0, s_n) = n$). Let P_n be the geodesic segment joining r_n and s_n. Since R and S diverge (exponentially) there is an N for which $l(P_n) > 2\delta$ whenever $n \geq N$. By passing to a subsequence if necessary and relabeling we may assume that $l(P_1) > 2\delta$ and the lengths of the P_n are strictly increasing as $n \to \infty$.

For each n in our index set (which is no longer all of $\mathbf{Z}^+$), there is a point $p_n \in P_n$ such that both $\mathrm{d}(p_n, R) \leq \delta$ and $\mathrm{d}(p_n, S) \leq \delta$. (Reason : Set $P_{nR} = \{p \in P_n : \mathrm{d}(p, R) \leq \delta\}$ and $P_{nS} = \{p \in P_n : \mathrm{d}(p, S) \leq \delta\}$. Using the δ-thin triangle with vertices $r_n 0 s_n$, every $p \in P_n$ is in at least one of the above sets. Thus the connected set P_n is the union of the closed sets P_{nR} and P_{nS}. They cannot be disjoint.) Let $r'_n \in \mathbf{p}_R(p_n)$ and $s'_n \in \mathbf{p}_S(p_n)$. By the triangle inequality, $\mathrm{d}(r'_n, s'_n) \leq 2\delta$. Using the (exponential) divergence of rays there are (bounded) initial segments R' and S' of R and S such that $\mathrm{d}(R \setminus R', S \setminus S') \geq 2\delta$, i.e. the points r'_n and s'_n lie in $R' \cup S'$ for all n. If d is the diameter of $R' \cup S'$ then $\mathrm{d}(0, p_n) \leq \mathrm{d}(0, r'_n) + \mathrm{d}(r'_n, p_n) < d + \delta$.

The above paragraph shows that each P_n meets the open ball around zero of radius $d + \delta$. In fact each P_n has a vertex in the closed ball of radius $d + \delta + 1$. Using the pigeonhole principle, infinitely many P_n share a common vertex v. Pass to a further subsequence so that each P_n passes through v. Now consider the sphere about v of radius 1. There will be at least two vertices on the sphere which intersect infinitely many of the P_n. Pass to the corresponding subsequence P_{n1}. Repeat the process for the sphere about v of radius $k = 2, 3, 4, \cdots$, to obtain further subsequences P_{nk}. Set $P = \bigcap_k \bigcup P_{nk}$.

By construction P is geodesic: if a and b are points of P then they lie on the geodesic segment P_n for some n. Also P is unbounded. Finally P^+ is in the 2δ-corridor about S: Let $y \in P^+$. Then $y \in P_n^+$ for some n. If $\mathrm{d}(y, s_n) < \delta$ we're done, so suppose not. By the thin triangle $s_n v s'_n$, either $\mathrm{d}(y, S) < \delta$ or $\mathrm{d}(y, \overline{v s'_n}) < \delta$. In the latter case

$$\mathrm{d}(y, S) \leq \mathrm{d}(y, s'_n) \leq \mathrm{d}(y, \overline{v s'_n}) + \mathrm{d}(v, s'_n) < \delta + \delta = 2\delta.$$

Similarly $P^- \sim R$. ∎

Definition 2.3: Recall that a subset S of a path metric space is *quasiconvex* (K) for some $K \geq 0$, if every geodesic segment α with $\alpha(0), \alpha(1) \in S$ satisfies $\sup_{t \in [0,1]} \mathrm{d}(\alpha(t), S) < K$.

Lemma 2.4. *Let $R \subset \Gamma$ be a ray and $r \in R$. If $a, b \in H(R, r)$ and $c \in \overline{ab}$, then $d(c, H(R, r)) < 2\delta$. (Half-spaces and complements of half-spaces are quasiconvex (2δ) .)*

Proof : Let $p \in \mathbf{p}_R(b)$ and $q \in \mathbf{p}_R(a)$. Consider the (2δ-thin) quadrilateral $abpq$. The point c must be within 2δ of $\overline{bp} \cup \overline{pq} \cup \overline{qa}$, and all of these segments lie in $H(R,r)$. A similar argument shows that $H^-(R,r)$ is also quasiconvex (2δ). ∎

Lemma 2.5. *Let R be a ray, $r = R(r)$ a point on R, and k a positive real number. Then the distance from $H^-(R,r)$ to $H(R,r+k\delta)$ exceeds $(k-12)\delta$. (Halfspaces are "thick".)*

Proof : We may assume that $k > 12$. Let $a \in H^-(R,r)$ and $b \in H(R,r+k\delta)$. Choose closest point projections $p \in \mathbf{p}_R(a)$, and $q \in \mathbf{p}_R(b)$ with the latter inside $H(R,r+k\delta)$. Set $p' = R(p+4\delta)$ and $q' = R(q-4\delta)$.
CLAIM: $\mathrm{d}(p', \overline{ap}) \geq 2\delta$. If not, there exists a point $t \in \overline{ap}$ with $\mathrm{d}(t,p') \leq 2\delta$. Since $p \in \mathbf{p}_R(t)$ (see remarks following 1.9), it follows that $\mathrm{d}(t,p) < \mathrm{d}(t,p') < 2\delta$. But then the triangle inequality says

$$4\delta = \mathrm{d}(p,p') \leq \mathrm{d}(p,t) + \mathrm{d}(t,p') < 2\delta + 2\delta = 4\delta$$

which is most certainly a contradiction. Therefore the claim holds.

By exactly the same argument, each of $\mathrm{d}(p', \overline{bq})$, $\mathrm{d}(q', \overline{bq})$, and $\mathrm{d}(q', \overline{ap})$ is at least 2δ. Using thin quadrilaterals (exercise 1.2) both p', q' are within 2δ of $\overline{ab}$. Let a', b' be respective closest point projections of p', q' into $\overline{ab}$. Then

$$\begin{aligned} k\delta \leq \mathrm{d}(p,q) &= 4\delta + \mathrm{d}(p',q') + 4\delta \\ &< 4\delta + 2\delta + \mathrm{d}(a',b') + 2\delta + 4\delta \\ &< 4\delta + 2\delta + \mathrm{d}(a,b) + 2\delta + 4\delta \\ &= 12\delta + \mathrm{d}(a,b). \end{aligned}$$

The conclusion follows. ∎

The statement and argument of lemma 2.4 need to be modified when the two endpoints are at infinity.

Lemma 2.6. *If $R \subset \Gamma$ is a ray, $r \in R$ and $L \subset \Gamma$ is a (geodesic) line with endpoints $L(+\infty)$ and $L(-\infty)$ both inside $D(R, r+14\delta)$, then $L \subset H(R,r)$.*

Proof : By hypothesis, there exist sub-rays L^+ and L^- of L entirely contained in $H(R,r+14\delta)$. Let $a \in L^+$ and $b \in L^-$. Then $\overline{ab}$ (the segment of L between a and b) stays within 2δ of $H(R,r+14\delta)$ by quasiconvexity. Lemma 2.5 implies that $H(R,r+14\delta)$ is more than 2δ from $H^-(R,r)$ and the result follows. ∎

The last result of this section is proved by a straightforward 2δ-thin quadrilaterals argument (see I.12 of [Sw]).

Exercise 2.7: Let X be negatively curved and $R \subset X$ a ray. If $r = R(r) \in R$ and $p \in H(R,r+8\delta)$, then $\mathrm{d}(p,R) < \mathrm{d}(p,H^-(R,r)) + 4\delta$.

3. Convergence and the Main Theorem

The boundary ∂X of a negatively curved space X was originally defined in terms of sequences of points in X convergent at infinity (see 1.8 of [Gr]). Here is a very intuitive definition of the latter phrase.

Definition 3.1: Let X be a negatively curved space, $\{a_n\} \subset X$ a sequence, and $R \subset X$ a (geodesic) ray. Define $a_n \to [R]$ to mean: given any $r (= R(r))$ there exists a positive integer N such that $a_n \in H(R, r)$ for all $n \geq N$.

The pointwise convergence at infinity of a sequence of functions $\{f_n\}$ on X now makes sense. (Since ∂X is a metric space, one needs no special definition of convergence for a sequence of points in ∂X.) Uniform convergence at infinity is defined similarly:

Definition 3.2: Let $S \subset X$, $\{f_n\}$ a sequence of functions each mapping X into X, and $R \subset X$ a ray. Define $f_n(x) \to [R]$ *uniformly on* S to mean: given any $r (= R(r))$ there exists a positive integer N such that $f_n(S) \subset H(R, r)$ for all $n \geq N$.

Theorem 3.3. *Let $R \subset \Gamma$ be a ray based at 0, and $\{g_m\}$ a sequence of distinct group elements. If $g_m(z_0) \to w = [R]$ for some point $z_0 \in \Gamma$, then $g_m(z) \to w$ for all $z \in \Gamma$. Furthermore, the convergence is uniform on compact subsets of Γ.*

Proof : Let $\epsilon > 0$ and let B denote the open ball with center z_0 and radius ϵ. Let $r \in R$ and choose $N > \frac{4\epsilon}{\delta} + 8$. By hypothesis there is some $M > 0$ such that $w_m = g_m(z_0) \in H(R, r + N\delta)$ for all $m \geq M$. Set $R^+ = R[r + N\delta)$ as the sub-ray of R from $r + N\delta$ onward. There are two possibilities.

$CASE\ 1 : \mathrm{d}(w_m, R^+) \geq 4\delta + \epsilon$ (The distance from w_m to R^+ is "large").
We know $w_m \in H(R, r + N\delta) \subset H(R, r + 8\delta)$. Using 2.7 we have

$$\mathrm{d}(w_m, R) - 4\delta < \mathrm{d}(w_m, H^-(R, r)), \quad \text{so}$$

$$\epsilon = (4\delta + \epsilon) - 4\delta \leq \mathrm{d}(w_m, R) - 4\delta < \mathrm{d}(w_m, H^-(R, r)).$$

Therefore $g_m(B) = B(w_m, \epsilon) \subset H(R, r)$.

$CASE\ 2 : \mathrm{d}(w_m, R^+) < 4\delta + \epsilon$ (The distance from w_m to R^+ is "small").
In this case, $g_m(B)$ is in the $4\delta + 2\epsilon$ neighborhood of R^+. Let $z \in g_m(B)$, so $\mathrm{d}(z, R^+) < 4\delta + 2\epsilon$. Suppose that $z \notin H(R, r)$. Then z must be closer to the segment $\overline{0r}$ than to R^+, in particular $\mathrm{d}(z, \overline{0r}) < 4\delta + 2\epsilon$. Then

$$N\delta = \mathrm{d}(\overline{0r}, R^+) \leq \mathrm{d}(\overline{0r}, z) + \mathrm{d}(z, R^+) < (4\delta + 2\epsilon) + (4\delta + 2\epsilon) = 8\delta + 4\epsilon,$$

contradicting our choice of $N > \frac{4\epsilon}{\delta} + 8$. Thus z must be in $H(R, r)$ and hence $g_m(B) \subset H(R, r)$.

Set $Z = \{z \in \Gamma : g_m(z) \to [R]\}$. By the above, Z is open. But Z is also closed: If $\{z_i\} \subset Z$ with $z_i \to z$, then for large i, $z \in B(z_i, \rho)$ for some fixed $\rho > 0$. By the previous paragraph, $z \in Z$, meaning $Z = \Gamma$. Uniform convergence on closed balls (hence on compact sets) is immediate. ∎

Theorem 3.4. *G acts as a convergence group on $\partial\Gamma$ (compare with 8.1.G of [Gr]).*

Proof : Suppose G is elementary. If $\partial\Gamma = \emptyset$ then the theorem is vacuously true. If $\partial\Gamma$ consists of two points, then every $g \in G$ either fixes or interchanges these points, and again the theorem is vacuously true. Assume that G is non-elementary.

Let $\{g_m\}$ be a sequence of distinct group elements and pick any vertex 0 as an origin. Without loss of generality we may assume that $0 \in \Gamma$ corresponds to the identity of G so that $g_m(0) = g_m \cdot id = g_m$, and that each g_m is a minimal representative as a word in the generating set of G. From a geometric viewpoint this means that g_m regarded as an edge path from 0 is a geodesic segment. By passing to a subsequence if necessary, we may suppose that $d(0, g_m) \geq 2m + 1$ for each $m \in Z^+$. Let $\mathcal{S}_1$ be the (finite) set of edges having 0 as a vertex. It is evident that infinitely many of the edge paths g_m pass through some edge $s_1 \in \mathcal{S}_1$. Pass to this corresponding subsequence $\{g_{1,m}\}$, and pick out and save a shortest element h_1 from this subsequence.

Let v_1 denote the other vertex of s_1. Let $\mathcal{S}_2$ be the collection of edges having v_1 as a vertex. Infinitely many of the edge paths $g_{1,m}$ pass through some edge $s_2 \in \mathcal{S}_2$ other than s_1. Let v_2 denote the other vertex of s_2 and pass to the corresponding subsequence $\{g_{2,m}\}$. Pick out and save a shortest word h_2 (distinct from h_1) from this new subsequence. Proceed recursively to obtain an edge path $S = s_1 s_2 s_3 \cdots$ and a diagonal subsequence $\{h_i\}$ of the original sequence. By construction, each path $\overline{0h_i}$ has an initial segment lying on S of length at least i.

Note that S, being a limit of geodesic segments, is a geodesic ray with initial point 0. Let $s \in S$. Then for large i, a shortest path from the vertex $h_i = h_i(0)$ to 0 passes thru s. This implies that $h_i \in H(S, s)$ for all sufficiently large i, i.e. $h_i(0) \to [S]$. Repeat the above construction with respect to the sequence $\{h_i^{-1}\}$ to obtain a geodesic ray T, and subsequence $\{g_n\}$ such that $g_n^{-1}(0) \to [T]$.

The strategy is to show that given any half-spaces $H(S, s)$ and $H(T, t)$ about S, T respectively, we can find an N such that $g_n(H^-(T, t)) \subset H(S, s)$ for all $n \geq N$. Since for any neighborhood U of $[S]$ and compact $K \subset \partial\Gamma \setminus [T]$ we can find s and t far enough from 0 so that $K \subset D^-(T, t)$ and $D(S, s) \subset U$, the above sentence implies $g_n \to [S]$ uniformly on K. Similarly, $g_n^{-1} \to [T]$ locally uniformly on $\partial\Gamma \setminus [S]$, establishing (CON).

We know that each g_n as a word (edge path) consists of an initial string $s_n \subset S$ of length n and that s_n is the initial part of s_{n+1}. Similarly g_n^{-1} has initial string $t_n \subset T$ of length n. Since the length of g_n is at least $2n + 1$ we know the end of s_n does not involve the start of t_n^{-1}, i.e. $g_n = s_n w_n t_n^{-1}$, where w_n is some word of length at least one. Now observe that $s_n w_n t_n^{-1}$ is a geodesic path implies that the segment $\overline{s_n g_n} = \overline{s_n(0) g_n(0)}$ lies inside $H(S, s_n)$.

Let $s \in S$. Choose N large enough so that both $s_N \geq s + 13\delta$ and $t_N \geq t + 15\delta$. Let $z \in H^-(T, t)$ and suppose by way of contradiction that $g_n(z) \in H^-(S, s)$ for some $n \geq N$. Let $q \in \mathrm{p}_S(g_n(z))$ and consider the triangle $g_n(0) q g_n(z)$. By choice of n, we know that $d(s_n, q) \geq 13\delta$. Using thickness of half-spaces (lemma 2.5) $d(s_n, \overline{q g_n(z)}) > \delta$. Therefore thin triangles says that

$$d(s_n, \overline{g_n(0) g_n(z)}) = d(g_n^{-1}(s_n), \overline{0z}) = d(t_n w_n^{-1}, \overline{0z}) < \delta.$$

However, the vertex $t_n w_n^{-1}$ lies in the half-space $H(T, t_n) \subset H(T, t + 15\delta)$. Using thickness of half-spaces, the distance from $H(T, t_n)$ to $H^-(T, t)$ is more than 3δ. Finally, since $H^-(T, t)$ is quasiconvex (lemma 2.4), we know that the path $\overline{0z}$ strays at most 2δ from $H^-(T, t)$. Hence

$$3\delta < \mathrm{d}(t_n w_n^{-1}, H^-(T,t)) \le \mathrm{d}(t_n w_n^{-1}, \overline{0z}) + 2\delta < \delta + 2\delta = 3\delta, \text{ a contradiction.}$$

Therefore $g_n(z) \in H(S,s)$ after all, and since z was arbitrary, g_n maps all of $H^-(T,t)$ into $H(S,s)$ as required. ∎

Recall that if G is a convergence group on a space Y, then the *limit set* $\Lambda(G)$ consists of all points $y \in Y$ such that there exists $x \in Y$ and a distinct sequence $\{g_n\} \subset G$ such that $g_n(x) \to y$. The ordinary set $\Omega(G)$ is the complement of the limit set. Since any proper compact subset of $\Gamma \cup \partial\Gamma$ is contained in some suitably large halfspace (along with the corresponding disk at infinity), the proof of 3.4 yields

Corollary 3.5. *G acts as a convergence group on $\overline{\Gamma} = \Gamma \cup \partial\Gamma$, with $\Omega(G) = \Gamma$ and $\Lambda(G) = \partial\Gamma$.* ∎

Gehring and Martin [GM] classify elements of convergence groups (acting on S^n) as *elliptic*, *parabolic*, or *loxodromic*. Elliptic elements are torsion, parabolics have a unique fixed point on S^n, and loxodromics have two fixed points on S^n. Similarly (see 8.1 of [Gr]), Gromov classifies the elements of any negatively curved group as either *elliptic* (=torsion) or *hyperbolic* (=non-torsion). He shows that a hyperbolic element has two fixed points on $\partial\Gamma$, one being attractive and the other repulsive. In light of 3.4 it is clear (at least whenever $\partial\Gamma = S^n$) that if G is negatively curved then $g \in G$ is hyperbolic if and only if g is loxodromic. I will use the term "loxodromic" exclusively hereafter.

Definition 3.6: Let G be a (discrete) convergence group acting on a compact metric space (X,d). We say that the limit point w is a *point of approximation* if there is associated with w a sequence $\{g_m\}$ of distinct group elements such that for each $x \in X \setminus \{w\}$ there is some $\epsilon = \epsilon(x)$ satisfying $\mathrm{d}(g_m(w), g_m(x)) \ge \epsilon$ for all m.

As an example, let G be a Kleinian group and $g \in G$ a loxodromic element. The fixed points of g on S^2 are both points of approximation. On the other hand, no parabolic fixed point can be a point of approximation [Ma]. Evidently, every loxodromic fixed point (in the boundary at infinity) of a negatively curved group is a point of approximation. In fact more is true.

Theorem 3.7. *Let G be a negatively curved group. Then every $x \in \partial\Gamma$ is a point of approximation (compare with 8.2.J in [Gr]).*

Proof : Let $x, y \in \partial\Gamma$ be distinct points and let L be any geodesic line with $L(+\infty) = x$ and $L(-\infty) = y$. Pick a vertex on L, call it v_0. Let L^+ denote that part of L between v_0 and x. Label the succesive vertices of L from v_0 tending towards x as v_1, v_2, v_3, $\cdots$. For each m, let $g_m \in G$ be the group element taking v_m to v_0. Use the convergence property to obtain a subsequence $\{g_k\}$ and rays S and T such that

$$g_k \to [S] \text{ locally uniformly on } \partial\Gamma \setminus [T] \quad \text{and}$$

$$g_k^{-1} \to [T] \text{ locally uniformly on } \partial\Gamma \setminus [S].$$

We can pass to a further subsequence so that $\{g_j([T])\}$ converges as well. Since $g_j^{-1}(v_0) \to [L^+]$ by construction, it is clear that both T and L^+ represent the point $x \in \partial\Gamma$. Because $y \in \partial\Gamma$ was chosen to be distinct from $x = [T]$ we know that $g_j(y) \to [S]$. Suppose for the moment that $g_j([T]) \to [S]$ also. Pick $s \in S$ so that $v_0 \notin H(S, s)$. Then for all sufficiently large j it is true that both $g_j(x) = g_j([T])$ and $g_j(y)$ are inside $D(S, s + 14\delta)$. Lemma 2.6 implies that the geodesic $g_j(L)$ is contained in $H(S, s)$. But then

$$v_0 = g_j(v_j) \in g_j(L) \subset H(S, s), \text{ a contradiction.}$$

Therefore $g_j(x)$ cannot converge to $[S]$. Since $g_j(w) \to [S]$ for all $w \neq x$, it is clear that x is a point of approximation. ∎

References

[C1] J. Cannon, The Combinatorial Structure of Cocompact Discrete Hyperbolic Groups, Geom. Dedicata **16** (1984), 123-148.

[C2] —, The Theory of Negatively Curved Spaces and Groups, in *Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces*, eds. T. Bedford, M. Keane, and C. Series, Oxford Univ. Press, 1991, 315-369.

[CE] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland Mathematical Library **9**, North-Holland, 1975.

[Ga] D. Gabai, Convergence groups are Fuchsian groups, Ann. of Math. **136** (1992), 447-510.

[GH] E. Ghys and P. de La Harpe, Infinite Groups as Geometric Objects (after Gromov), in *Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces*, eds. T. Bedford, M. Keane, and C. Series, Oxford Univ. Press, 1991, 299-314.

[GM] F. W. Gehring and G. Martin, Discrete Quasiconformal Groups I, Proc. London Math. Soc. **55** (1987), 331-358.

[Gr] M. Gromov, Hyperbolic Groups, in *Essays in Group Theory*, ed. S. M. Gersten, MSRI series **8** (1987), 75-263.

[Ma] B. Maskit, *Kleinian Groups*, Grundlehren der mathematischen Wissenschaften **287**, Spinger-Verlag, 1988.

[MS] G. Martin and R. Skora, Group Actions of the 2-Sphere, Amer. J. Math. **111** (1989), 387-402.

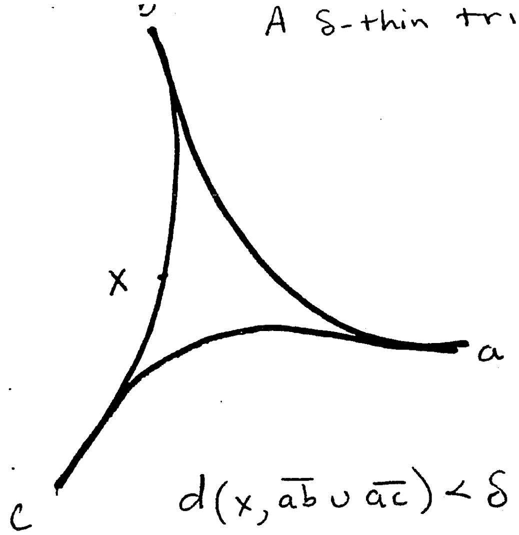
[O] A. Yu. Ol'shanskii, Almost Every Group is Hyperbolic, Internat. J. Algebra Comput. **1** (1992), 1-17.

[S] Z. Sela, On the Isomorphism Theorem for Hyperbolic Groups and the Classification of 3-Manifolds, IHES preprint.

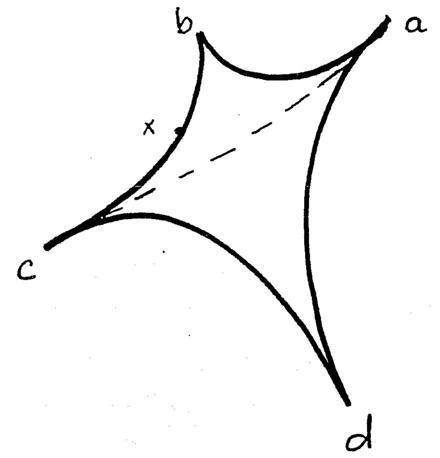
[Sw] E. L. Swenson, Negatively Curved Groups and Related Topics, Thesis, Brigham Young University, 1993.

[T] P. Tukia, Homeomorphic Conjugates of Fuchsian Groups, J. für Reine und Agnew. Math. **391** (1988), 1-54.

A δ-thin triangle

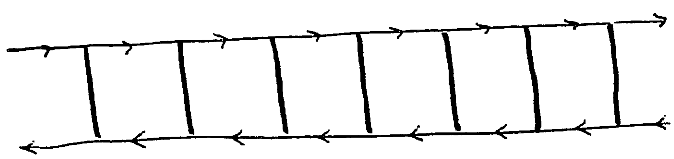


$d(x, \overline{ab} \cup \overline{ac}) < \delta$

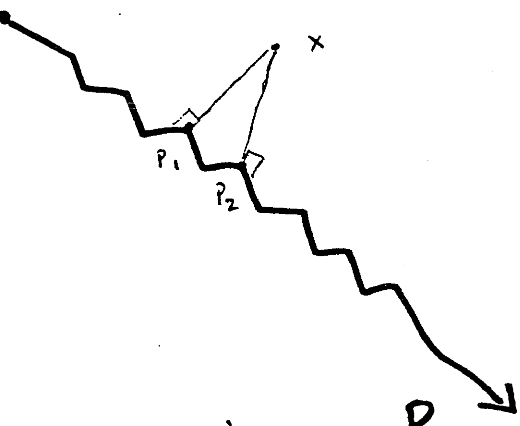


Quadrilaterals are 2δ-thin

Cayley graph for $\langle a, b \mid (ab)^2 = b^2 = 1 \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2$

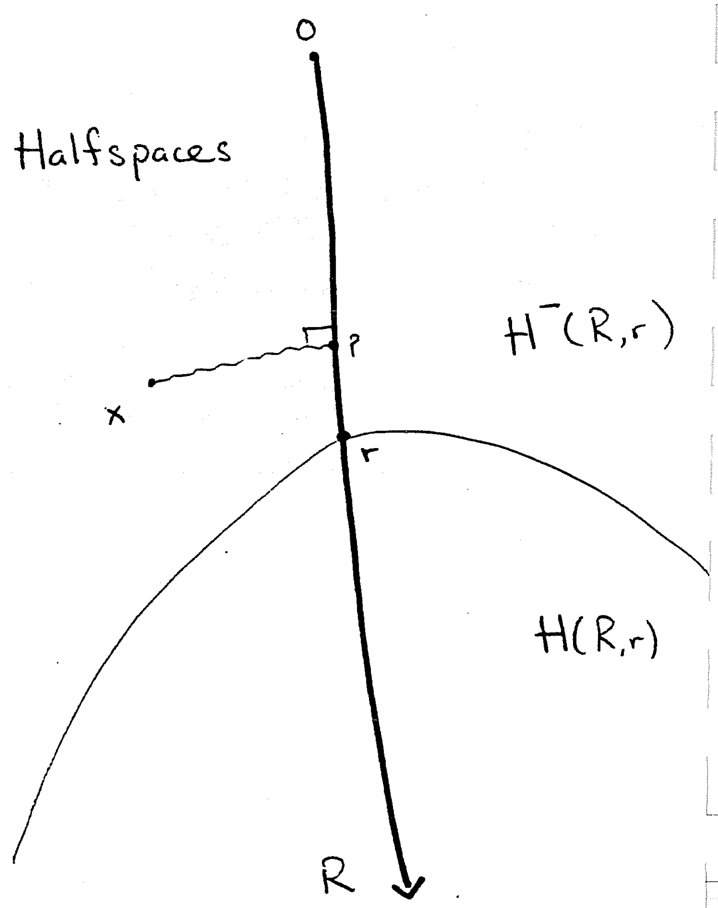


($\delta = 1+\varepsilon$, two pts. at infinity)

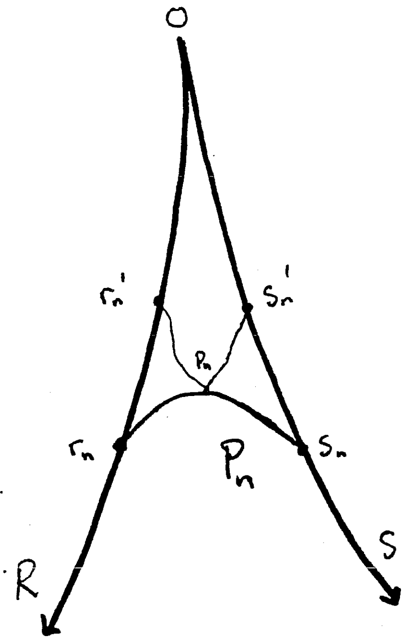


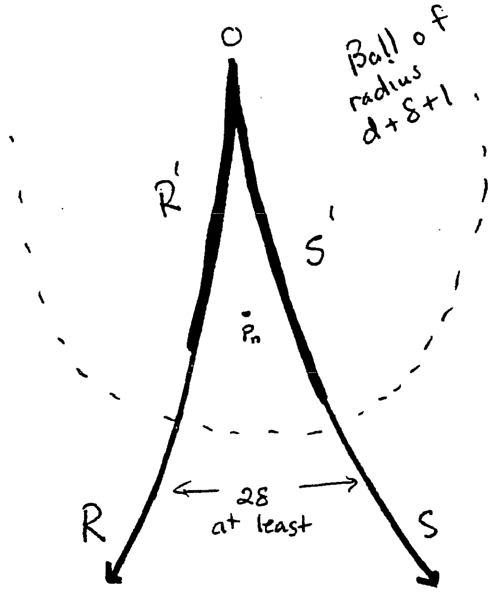
losest point
rojection is
a relation

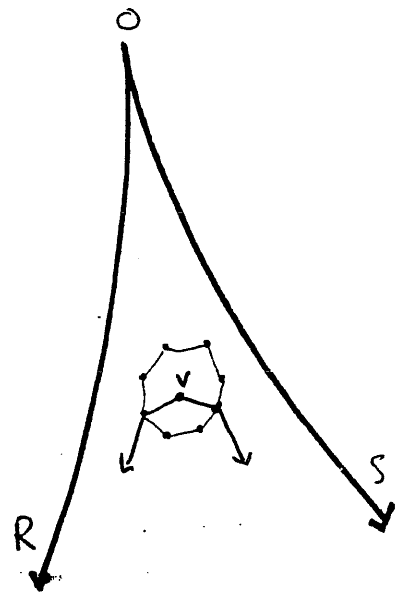
Halfspaces

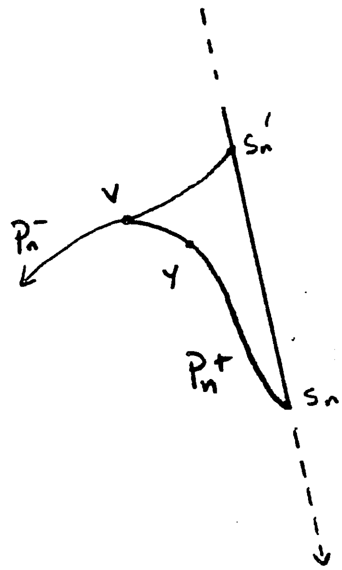


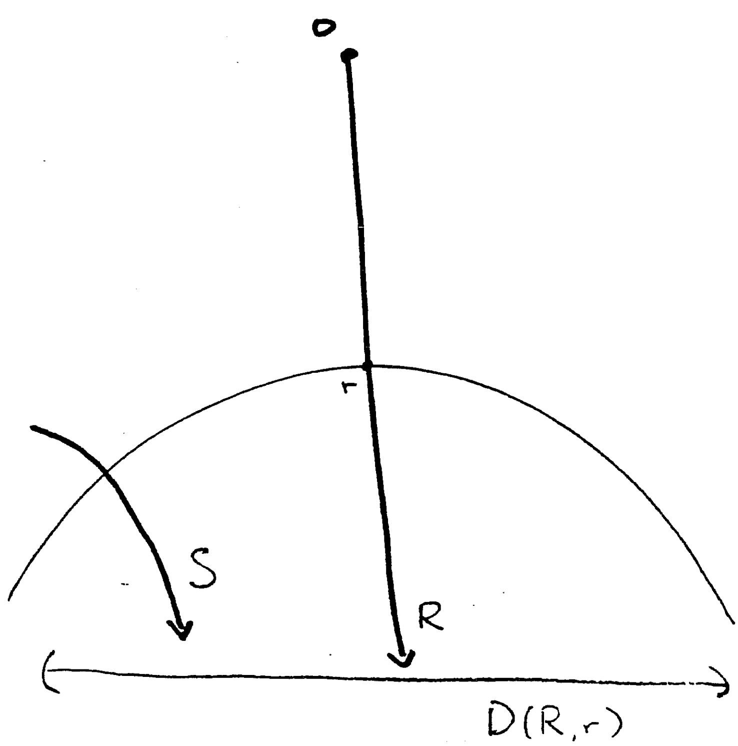
Lemma 2.2





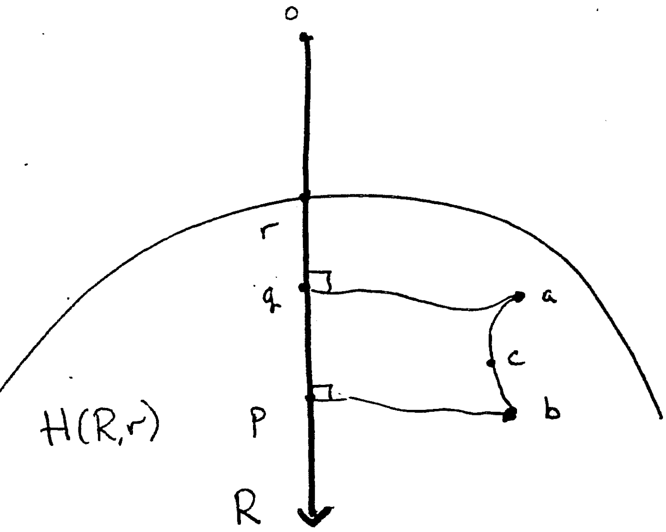




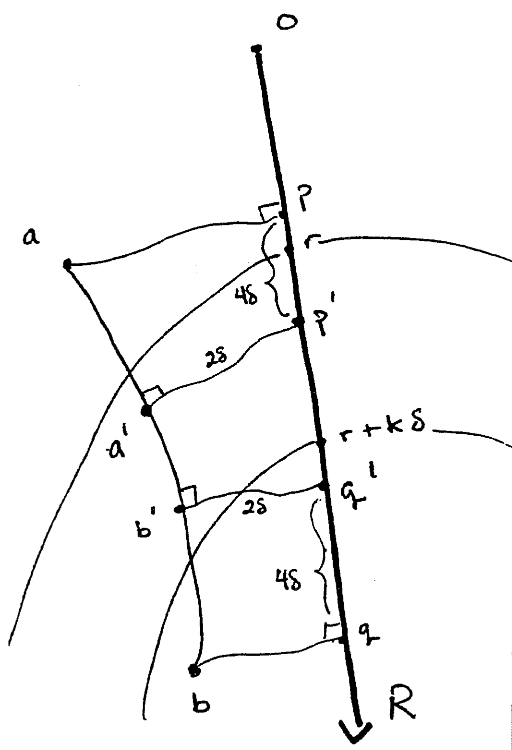


Disk at infinity; note that

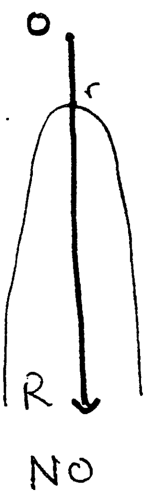
Halfspaces are quasiconvex (2δ)

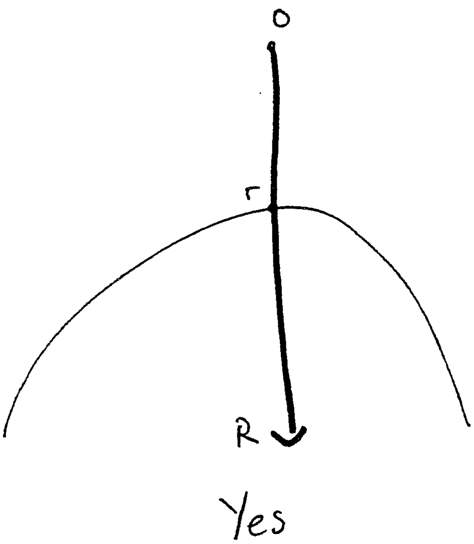


Lemma 2.5

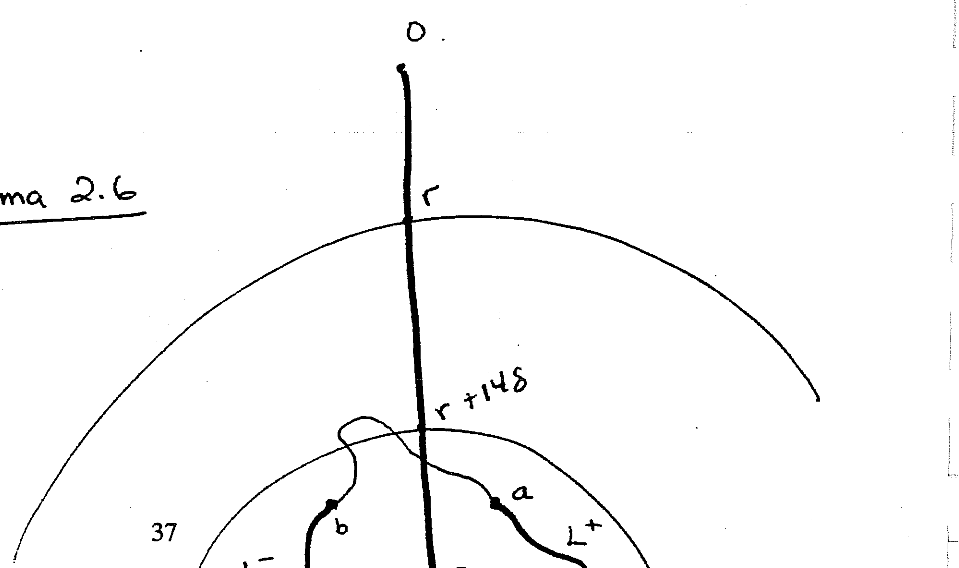


Halfspaces are "thick"

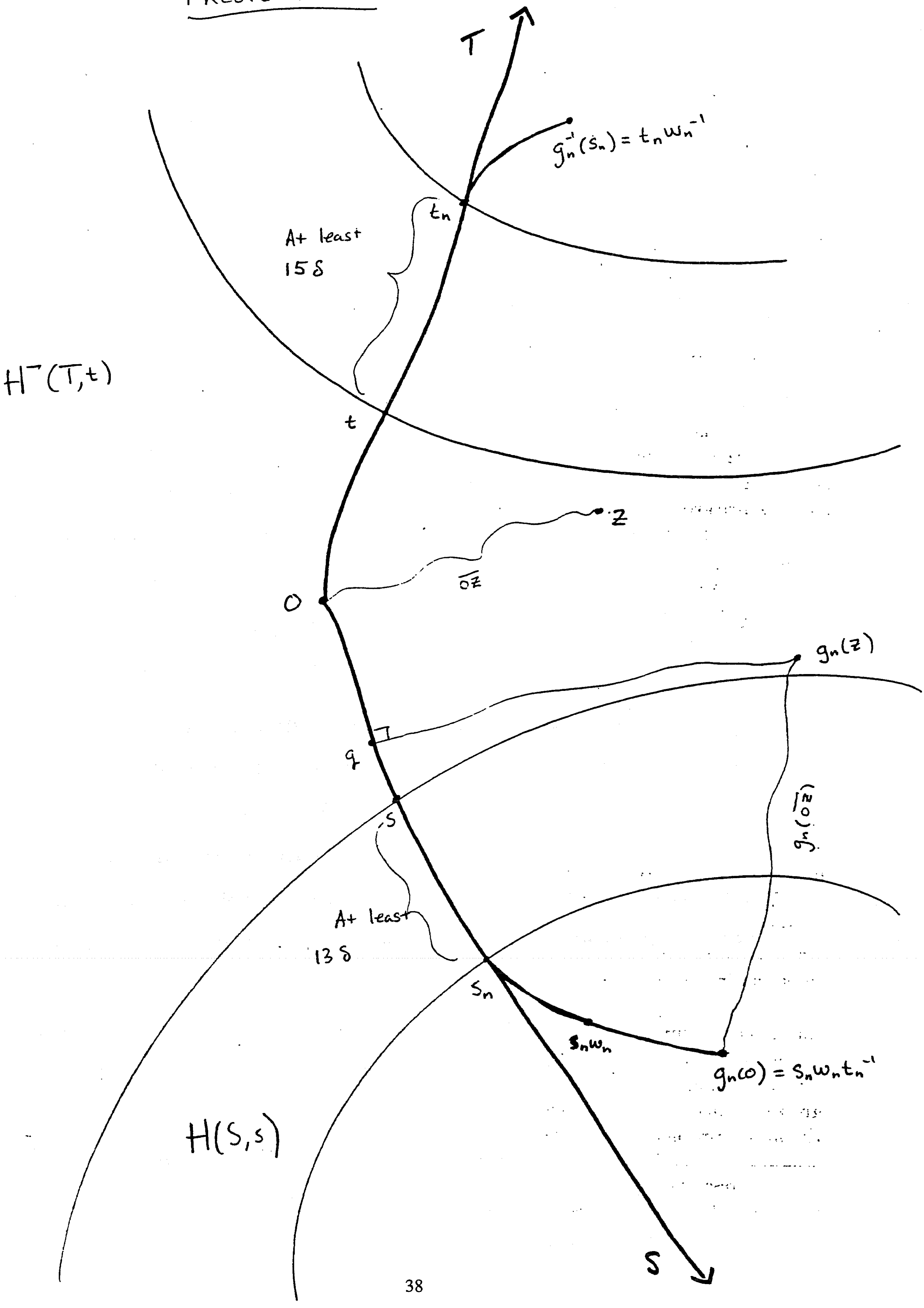




Lemma 2.6



Theorem 3.4


T
$g_n^{-1}(s_n) = t_n w_n^{-1}$
t_n
At least 15δ
$H^-(T,t)$
t
z
$\overline{oz}$
o
$g_n(z)$
q
s
$g_n(\overline{oz})$
At least 13δ
s_n
$s_n w_n$
$g_n(o) = s_n w_n t_n^{-1}$
$H(S,s)$
S

Some of Daverman's Wild Strongly Homogeneous Cantor Sets are Slippery

Craig R. Guilbault

Abstract. Problem #7 from the Proceedings of the 1986 Western Workshop in Geometric Topology asks whether the Daverman wild Cantor sets are slippery. We apply recent results on spines of compact contractible manifolds to show that in many cases the answer is yes.

1. Background

If M^n is a PL manifold, A is a polyhedron contained in $int(M^n)$, and M^n collapses to A; we say A is a *PL spine* of M^n. The following questions about a class of compact contractible 4-manifolds constructed by B. Mazur [Ma] remain open.

Question 1. *Do any Mazur 4-manifolds contain a pair of disjoint spines?*

Question 2. *Do all Mazur 4-manifolds contain a pair of disjoint spines?*

In [Gu] the high dimensional analogs of these questions are addressed. It is shown that there is a class of compact contractible manifolds in dimensions ≥ 9 which are not homeomorphic to a ball and which contain disjoint pairs of spines. In particular, it is shown that if a compact contractible n-manifold M^n may be realized as $S^n - int(N)$, where N is a regular neighborhood an acyclic k-complex in S^n with $4k<n$, then M^n contains disjoint spines. Since a compact contractible n-manifold of dimension at least 4 is uniquely determined by its boundary, this is equivalent to requiring that ∂M^n be homeomorphic to ∂N where again N is the regular neighborhood of an acyclic k-complex in S^n with $4k<n$.

At least part of the interest in these questions stems from possible connections to embedding theory. For example, there is belief that a sufficient understanding of these spines might yield an answer to the following long standing question from embedding theory.

Question 3. *Are all Cantor sets in S^n slippery?*

A set $K \subset S^n$ is *slippery* if for any $\varepsilon>0$, there exists an ε-homeomorphism $h: S^n \to S^n$ with $h(K) \cap K = \emptyset$. It is known that for $n \leq 3$ all Cantor sets in S^n are slippery.

In [Da], Daverman constructed an interesting collection of Cantor sets in S^n, which by virtue of their extreme wildness, are considered likely candidates for counterexamples to Question 3--if any counterexample exists. In [Gu], it is stated without proof that some of Daverman's wild Cantor sets are slippery. Here we provide the details to this observation.

2. The Daverman Construction

Let M^{n-2} be a compact acyclic (i.e. having the homology of a point) but not simply connected (n-2)-manifold containing a PL spine X' of dimension $k \leq n-3$. Let $N^{n-1} = M^{n-2} \times [-1,1]$ and $X = X' \times \{0\}$. Then define $Q^n = N^{n-1} \times [-2,2]$ and let G be the upper semicontinuous decomposition of Q^n having $\{X \times \{c\} \mid c \in C\}$ as its collection of nondegenerate elements, where C denotes the standard middle thirds Cantors set in $[0,1] \subset [-2,2]$. Let Q^* denote the decomposition space Q/G and $\pi: Q \to Q^*$ the decomposition map. Notice that, since X is not simply connected, our decomposition is not cell-like. Nevertheless, Q^* is a finite dimensional homology manifold which, by [BL] (or more recent generalizations), admits a cell-like resolution. (As an exercise, produce this resolution explicitly.) By verifying the disjoint disks property and applying [CBL] or [Ed] one may conclude that Q^* is an n-manifold. This argument may be found in [Da]. Additionally, note that Q^* is contractible--again see [Da]. Let $K \subset Q^*$ denote the image of the nondegeneracy set of π. Since $\partial Q \approx \partial Q^*$ we may form the manifold $Q \cup_{\partial} Q^*$ which is easily seen to be a homotopy n-sphere--hence, homeomorphic to S^n by the Generalized Poincaré Conjecture. Now, $K \subset Q^* \subset S^n$ is a Daverman wild Cantor set.

3. Slippery Wild Cantor Sets

We say that a Cantor set $K \subset S^n$ is "of type $\mathscr{C}$" if $K = \cap N_i$ where each N_i is a disjoint union of finitely many compact contractible n-manifolds and $N_{i+1} \subset int(N_i)$ for each i. If this defining sequence can be chosen so that each of the compact contractible n-manifolds used contains a pair of disjoint spines, we say that K is of type $\mathscr{C}_d$. For example, since a tame Cantor set in S^n can be defined as a nested intersection of n-balls, it is of type $\mathscr{C}_d$.

PROPOSITION 1. *If a Cantor set $K \subset S^n$ is of type $\mathscr{C}_d$ then K is slippery.*

Proof. Write $K = \cap N_i$ where each $N_i = M_{i_1} \cup M_{i_2} \cup \cdots \cup M_{i_k}$ (a finite disjoint union of compact contractible n-manifolds containing disjoint pairs of spines), and let $\varepsilon > 0$. Choose i sufficiently large that each M_{i_j} has diameter $<\varepsilon$. Observe that there is a homeomorhpism $h_{i_j}: M_{i_j} \to M_{i_j}$ which fixes ∂M_{i_j} and moves $K \cap M_{i_j}$ off itself. Indeed, since $K \cap M_{i_j} \subset int(M_{i_j})$ we may apply the techniques of Lemma 2.3 of [Gu] to obtain a spine A of M_{i_j} disjoint from $K \cap M_{i_j}$. Then use the collar structure on $M_{i_j} - A$ to "slide $K \cap M_{i_j}$ off itself". This is the desired h_{i_j}. Define $h: S^n \to S^n$ to be h_{i_j} on M_{i_j} and the identity on $S - N_i$. Clearly, h is an ε-homeomorphism and $h(K) \cap K = \emptyset$. □

PROPOSITION 2. *Each Daverman wild Cantor set is of type $\mathscr{C}$.*

Proof. Using the notation established in §1, consider $Q^n = N^{n-1} \times [-2,2]$. If $\delta = 1/9$ then $[-\delta, 1/3 + \delta] \cup [2/3 - \delta, 1+\delta]$ contains the middle thirds Cantor set and $N^{n-1} \times [-\delta, 1/3 + \delta]$ and $N^{n-1} \times [2/3 - \delta, 1+\delta]$ are disjoint copies of Q^n whose union contains G. Let Q_0 and Q_1 be slightly shrunken copies of these two sets so that $Q_0 \cup Q_1 \subset int(Q)$. Notice that the pairs $(Q_0, G|Q_0)$ and $(Q_0, G|Q_1)$ are equivalent to (Q^n, G), hence $Q_0^* = \pi(Q_0)$ and $Q_1^* = \pi(Q_1)$ are each compact contractible n-manifolds homeomorphic to Q^*. Iterating this process allows us to write K as $\cap N_i$ where $N_1 = Q^*$, $N_2 = Q_0^* \cup Q_1^*$, and in general N_i is a disjoint union of copies of Q^*. □

PROPOSITION 3. *Some Daverman wild Cantor sets are of type* $\mathscr{C}_d$. *Hence, they are slippery.*

Proof. By Proposition 1 and the proof of Proposition 2, we need only to identify a Daverman wild Cantor set $K \subset S^n$ for which the associated compact contractible n-manifold Q^* contains a disjoint pair of spines. By [Gu] (see earlier remarks), it suffices to find a Q^* such that ∂Q^* is realizable as the boundary of the regular neighborhood of an acyclic k-complex in S^n (4k<n). This is often the case. For example, one may begin the construction described in §1 by choosing M^{n-2} to be a regular neighborhood of such a k-complex L embedded in S^{n-2}. Now the canonical inclusion of S^{n-2} into S^n yields an embedding of L into S^n with regular neighborhood homeomorphic to $M^{n-2} \times I^2$ which is homeomorphic to Q^n; and which, by construction, has boundary homeomorphic to Q^*. □

REFERENCES

[Da] R.J. Daverman, *Embedding phenomena based upon decomposition theory: Wild Cantor sets satisfying strong homogeneity properties*, Proc. Amer. Math. Soc. 75(1979), 177-182.

[BL] J.W. Bryant, and R.C. Lacher, *Resolving zero-dimensional singularities in generalized manifolds*, Math. Proc. Cambridge Philos. Soc 83(1978), 403-413.

[CBL] J. Cannon, J.W. Bryant, and R.C. Lacher, *The structure of generalized manifolds having nonmanifold set of trivial dimension.* in "Geometric Topology" (J.C. Cantrell, ed.), Academic Press, New York, 1979.

[Ed] R.D. Edwards, *The topology of manifolds and cell-like maps*, Proc. International Congress of Mathematicians, Helsinki, 1978 (O. Lehti, ed.) Acad. Sci. Fenn. 1980, Helsinki, 111-127.

[Gu] C.R. Guilbault, *Some compact contractible n-manifolds containing disjoint spines*, Topology, to appear.

[Ma] B. Mazur, *A note on some contractible 4-manifolds*, Ann. of Math. (2) 73(1961), 221-228.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF WISCONSIN-MILWAUKEE
MILWAUKEE, WI 53201

Graphing Inverse Limits of Interval Maps

Richard M. Schori
Oregon State University

October 4, 1994

I. Introduction

Our goal is to graph approximations to the inverse limits of certain classes of unimodal (one critical point) and bimodal (two critical points) maps on the closed interval I.

We first show how to represent an inverse sequence in a more geometric way as a sequence of graphs and projection maps and then for certain classes of maps we will show how to embed these graphs in the plane in such a way that the graphs will approximate the inverse limits in the sense that the embeddings in the plane will converge in the Hausdorff metric to an embedding of the inverse limit. This will provide the framework for a method for approximating the inverse limit with computer graphics.

The techniques use algorithms on kneading sequences of pre-critical points of f. The author has code written in SCHEME, a dialect of LISP, that does the appropriate computations and then draws, on demand, the graph of different stages of the approximation.

This is a short version of a longer paper [S] that has all the technical details. I want to thank Joe Christy, Beverly Diamond, and Ethan Coven for help and encouragement.

Karen Brucks and Beverly Diamond [BD] have a fine contribution to this theory which apparently is more theoretical than the work here but has more restrictions in the computer applications.

II. Preliminary Definitions

If $f: I \to I$ is a continuous function on the closed unit interval, then

$$I \xleftarrow{f} I \xleftarrow{f} I \xleftarrow{f} \cdots$$

is the *inverse sequence* of f with associated inverse limit space inv(I, f) defined by

$$\text{inv}(I, f) = \{ \underline{x} = (x_0, x_1, x_2, \dots) : x_n \in I \text{ and } f(x_n) = x_{n-1} \text{ for } n \geq 1\}$$

with metric d given by

$$d(\underline{x}, \underline{y}) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

We are going to replace the above usual definition with an equivalent but more geometric version. Let

$$G_n = \{(f^n(x), f^{n-1}(x), \ldots, f(x), x) : x \in I\},$$

define $\alpha_n : I - G_n$ by

$$\alpha_n(x) = (f^n(x), \ \ldots, f(x), x),$$

and let $p_n : G_n \to G_{n-1}$ be the projection

$$p_n(f^n(x), \ \ldots, f(x), x) = (f^n(x), \ \ldots, f(x)).$$

Thus G_n is a graph in (n+1)-space, α_n is a homeomorphism, and p_n is a projection map that mimicks the action of f. We have

$$\begin{array}{ccccccc}
I & \xleftarrow{f} & I & \xleftarrow{f} \cdots & I & \xleftarrow{f} & I \leftarrow \cdots \\
\downarrow \alpha_0 & & \downarrow \alpha_1 & & \downarrow \alpha_{n-1} & & \downarrow \alpha_n \\
G_0 & \xleftarrow{p_1} & G_1 & \xleftarrow{p_2} \cdots & G_{n-1} & \xleftarrow{p_n} & G_n \leftarrow \cdots
\end{array}$$

where $p_n \circ \alpha_n = \alpha_{n-1} \circ f$. Thus, we have an induced map

$$\hat{\alpha} : \mathrm{inv}(I, f) \to \mathrm{inv}(G_n, p_n)$$

which is a homeomorphism. See Fig. 1 to illustrate p_1 and p_2 for $m = 2$.

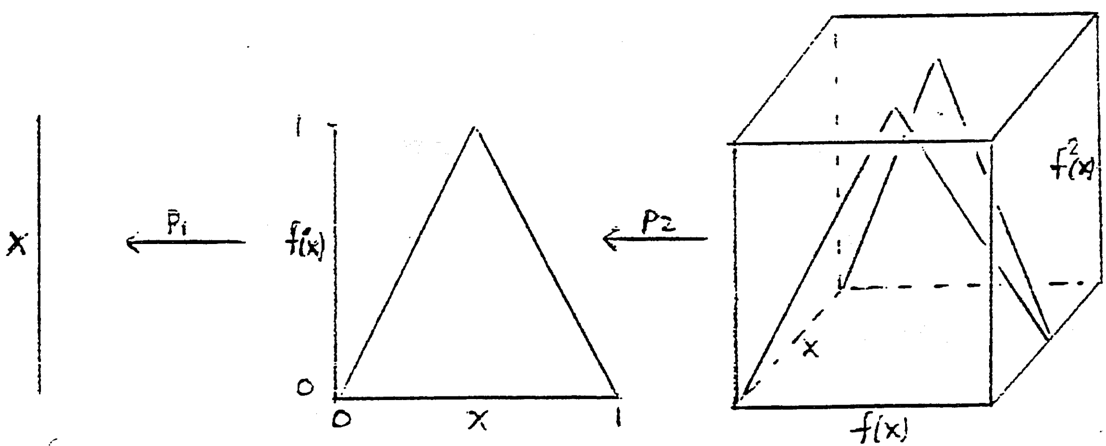


Fig. 1

It is the inverse sequence of the (G_n, p_n) that we regard as a more geometric version of the original inverse sequence of the (I, f).

In summary, we construction subsets $D_n \subset \mathfrak{R}^2$, homeomorphisms $\beta_n : G_n \to D_n$, and maps $q_n : D_n \to D_{n-1}$, as in the diagram below, so that a homeomorphism

$$\hat{\beta} : \mathrm{inv}(G_n, p_n) \to \mathrm{inv}(D_n, q_n)$$

is induced,

$$\begin{array}{ccccccccc}
G_0 & \xleftarrow{p_1} & G_1 & \xleftarrow{p_2} \cdots & G_{n-1} & \xleftarrow{p_n} & G_n & \leftarrow \cdots & \mathrm{inv}(G_n, p_n) \\
\downarrow\beta_0 & & \downarrow\beta_1 & & \downarrow\beta_{n-1} & & \downarrow\beta_n & & \downarrow\hat{\beta} \\
D_0 & \xleftarrow{q_1} & D_1 & \xleftarrow{q_2} \cdots & D_{n-1} & \xleftarrow{q_n} & D_n & \leftarrow \cdots & \mathrm{inv}(D_n, q_n)
\end{array}$$

and the maps q_n are so defined that the sets D_n converge in the Hausdorff metric to a set $D \subset \Re^2$ that is homeomorphic to $\mathrm{inv}(I, f)$.

III. The Tent Map Family

Here we outline the constructions for the tent map family which illustrates our techniques. All the following definitions depend on m and it is understood that they are all defined for a fixed m. Define

$$f(x) = \begin{cases} mx, & \text{if } 0 \le x \le .5 \\ m(1-x), & \text{if } .5 \le x \le 1 \end{cases}$$

and for writing code, define

$$f1(x) = \tfrac{x}{m}$$

which is the inverse function of $f|[0,.5]$. This gives us a method for finding the set $f^{-1}(x) = f^{-1}(\{x\}) = \{f1(x), 1 - f1(x)\}$, in case $0 \le x < f(.5)$.

Lemma 1. Turning points of $G_n = \alpha_n$(critical points of f^n)

Let $C(n) =$ critical points of f^n.

Itineraries and Kneading Sequences.

For $f{:}I \to I$ and $p \in I$, let the *itinerary* of p,

$$\mathfrak{I}(p) = M_0 M_1 M_2 \cdots$$

where $$M_i = \begin{cases} L, & \text{if } f^i(p) < .5 \\ R, & \text{if } f^i(p) > .5 \\ C, & \text{if } f^i(p) = .5\,. \end{cases}$$

Define the *restricted itenerary* of $p \in I$, denoted $k(p)$, to be the itinerary of p if C does not occur, but if it does occur, stop after the C. For example, when $m = 2$,

$k(.25) = \mathrm{LC}$, $k(.5) = \mathrm{C}$, $k(.75) = \mathrm{RC}$, and $k(.125) = \mathrm{LLC}$.

Let $\mathrm{K(n)} \equiv \{k(\mathrm{p}) : \mathrm{p} \in \mathrm{C(n)}\}$, and note that the map

$$k : \mathrm{C(n)} \to \mathrm{K(n)}$$

is 1-1. The inverse map

$$\mathrm{e} : \mathrm{K(n)} \to \mathrm{C(n)}$$

can also be easily defined. For example, $\mathrm{e(LC)} = .25$. Now, let

$$\mathrm{KNEAD(n)} \equiv \text{kneading sequences of ordered critical values of } \mathrm{f^n}.$$

Then $\mathrm{KNEAD(n)}$ is the desired symbolic representation for the turning points of $\mathrm{G_n}$ and will be used to develop algarithms for embedding $\mathrm{G_n}$ in the plane.

Example, for $\mathrm{m} = 2$, $\mathrm{n} = 2$,

$$\mathrm{C(2)} = \{.25.\ \ .5,\ \ .75\} \quad \text{and} \quad \mathrm{KNEAD(2)} = \{\mathrm{LC},\ \mathrm{C},\ \mathrm{RC}\}.$$

Now let $\mathcal{LR} =$ all finite L-R sequences.

CONVERTING L-R SEQUENCES TO REALS.

We need to embed the finite L-R sequences in the plane; as a composition of three maps. The conversion of L-R sequences to 0-1 sequences is automatic using the rule $\mathrm{L} \to 0$ and $\mathrm{R} \to 1$. We name this map (LR-to-SKEW).

Convert skew code to binary code with the map (SKEW-to-BINARY) which takes

$$\mathrm{a_1 \cdots a_n} \to \mathrm{b_1 \cdots b_n} \quad \text{where } \mathrm{b_j} = \sum_{i=1}^{j} \mathrm{a}_i \ (\mathrm{mod}\ 2).$$

Now convert binary code to reals with the map (BINARY-to-REAL) which takes

$$\mathrm{b_1 \cdots b_n} \to \sum_{\mathrm{i=1}}^{\mathrm{n-1}} \frac{2\mathrm{b}_i}{3^i} + \frac{\mathrm{b}_n}{3^{n-1}}.$$

Our final map $\mathrm{r} : \mathcal{LR} \to [0, 1]$ is the composition

$$\mathrm{r} = \text{(BINARY-to-REAL)} \circ \text{(SKEW-to-BINARY)} \circ \text{(LR-to-SKEW)}.$$

Illustrating the Embedding Algorithm.

We will discuss the example where $\mathrm{m} = 2$. Here, $\mathrm{KNEAD(1)} = \mathrm{(C)}$ says that the only turning point of $\mathrm{G_1}$ is $\alpha_1(\mathrm{e(C)}) = (\mathrm{f}(.5)\ \ .5)$. In this case, $\mathrm{G_1}$ is just the graph of f, $\mathrm{G(f^1)}$. We want to find the symbolizm that will generate the geometry as indicated here in Fig. 2 .

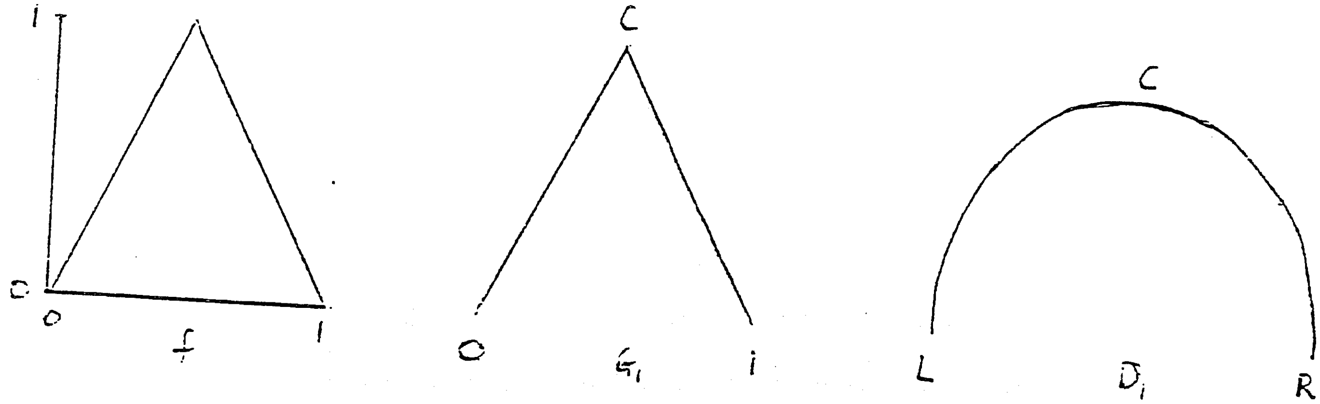


Fig. 2.

In particular, think of the turning point C connecting 0 and 1 in G_1 or C connecting L and R in D_1. Correspondingly, for

$$\mathrm{KNEAD}(2) = (\mathrm{LC}\ \ \mathrm{C}\ \ \mathrm{RC})$$

and $\mathrm{LL} \xrightarrow{r} 0$, $\mathrm{LR} \xrightarrow{r} \frac{1}{3}$, $\mathrm{RR} \xrightarrow{r} \frac{2}{3}$, and $\mathrm{RL} \xrightarrow{r} 1$, we want the turning point LC in G_2 to connect LL and RL in D_2, C in G_2 to connect RL and RR, and RC in G_2 to connect RR and LR.

Symbolically, we can write, for

D_1: $\mathrm{L} \xrightarrow{\mathrm{C}} \mathrm{R}$, and for

D_2: $\mathrm{LL} \xrightarrow{\mathrm{LC}} \mathrm{RL} \xrightarrow{\mathrm{C}} \mathrm{RR} \xrightarrow{\mathrm{RC}} \mathrm{LR}$,

and call these the *defining diagram* for the corresponding D_i. See Fig. 3 to illustrate f_2, G_2, and D_2 in the case that $m = 1.5$.

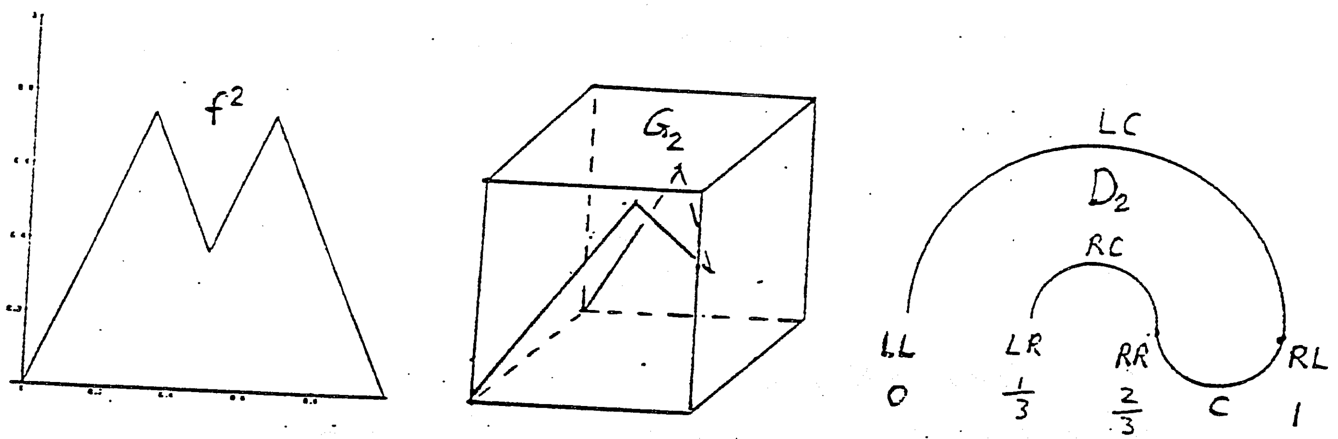


Fig. 3.

At the next stage we have

$$\mathrm{KNEAD}(3) = (\mathrm{LLC\ \ LC\ \ LRC\ \ C\ \ RRC\ \ RC\ \ RLC})$$

and the corresponding defining diagram for D_3 is

$$\mathrm{LLL} \xrightarrow{\mathrm{LLC}} \mathrm{RLL} \xrightarrow{\mathrm{LC}} \mathrm{RRL} \xrightarrow{\mathrm{LRC}} \mathrm{LRL} \xrightarrow{\mathrm{C}} \mathrm{LRR} \xrightarrow{\mathrm{RRC}} \mathrm{RRR} \xrightarrow{\mathrm{RC}} \mathrm{RLR} \xrightarrow{\mathrm{RLC}} \mathrm{LLR}.$$

We point out here that in the case there is one letter, C, corresponding to the turning point, the extracted diagram from above,

$$\mathrm{LRL} \xrightarrow{\mathrm{C}} \mathrm{LRR}$$

can be viewed as meaning that C acts on LRL, and since there is one letter in C, then the first letter from the right in LRL is changed, resulting in LRR. This is then interpreted as LRL being connected to LRR. Note that $\mathrm{LRL} \xrightarrow{\mathrm{I}} \frac{1}{3}$ and $\mathrm{LRR} \xrightarrow{\mathrm{I}} \frac{2}{9}$. If there are two letters, as in LC, then the second letter from the right in RLL is change resulting in RRL, etc.

In this short paper we omit the technical aspects of the longer paper [S] which gives the algorithms on which the computer code is based and which lay the foundation for the proof of the following which is the main theorem of the paper.

Theorem. The sets D_n in the plane converge in the Hausdorff metric to a set that is an embedding of inv(I, f) in the plane.

SAMPLES OF COMPUTER GRAPHICS.

We now show on the next few pages a few samples of the pictures that have been generated from the various versions of code that have been written for different maps.

References.

[BD]. Karen Brucks and Beverly Diamond, *A Symbolic Representation of Inverse Limit Spaces for a Class of Unimodal Maps,* manuscript.

[S] Richard M. Schori, *Mathematics of Graphing Inverse Limits of Interval Maps,* manuscript.

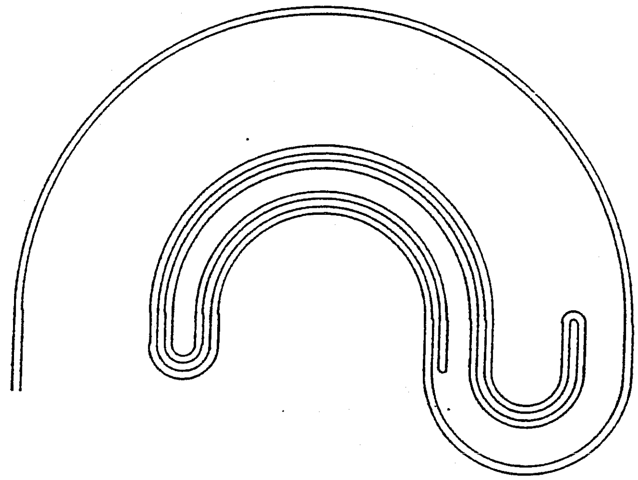
I. TENT MAPS.

a) For $0 \leq m < 1$, the inverse limit, LIM is a point.

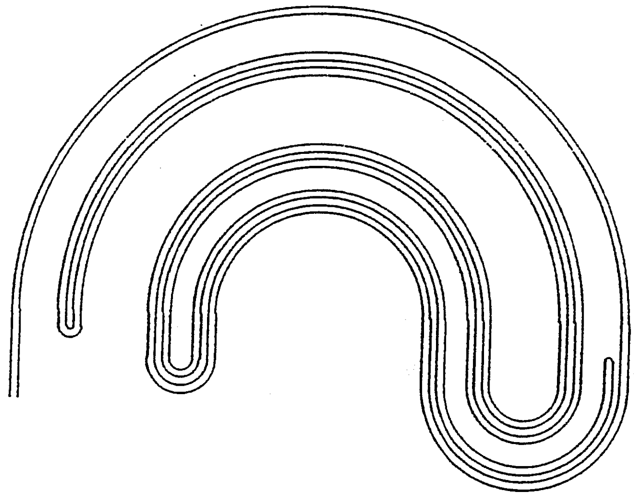
b) For $m = 1$, LIM is an arc.

c) For $m > 1$, LIM contains an indecomposable continuum.

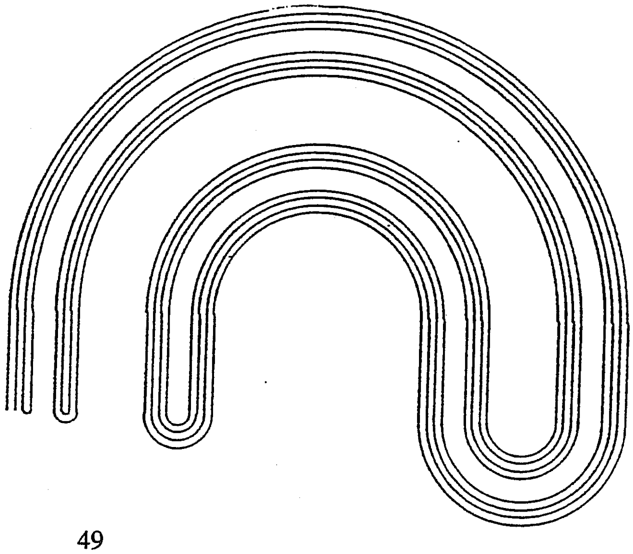
For $m = 1.512$, $D_5 =$



For $m = 1.722$, $D_5 =$



For $m = 2$, $D_5 =$

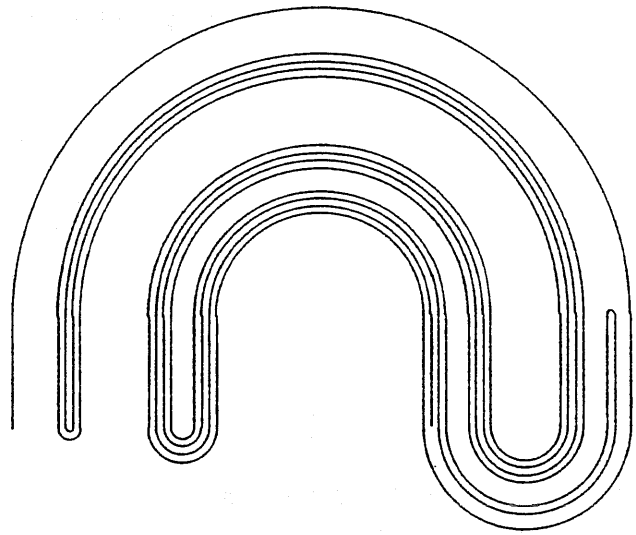


II. Topologist's map.

PL-maps through (0,0). (.5,1), and (1,t), for $t \in I$.

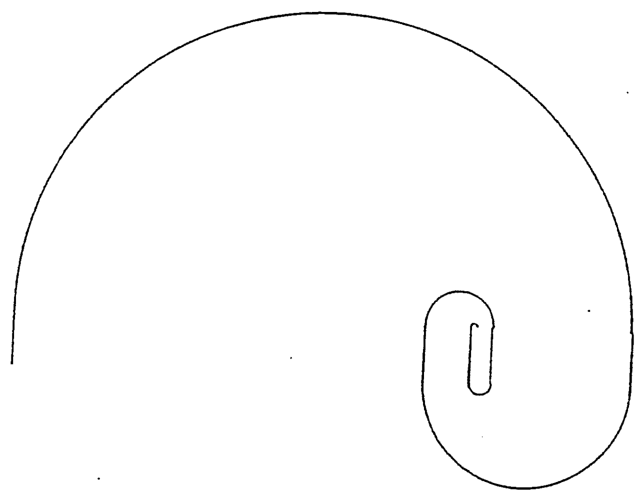
a) For $0 \leq t < .5$, LIM contains an indecomposable continuum.

For $t = .2$, $D_5 =$



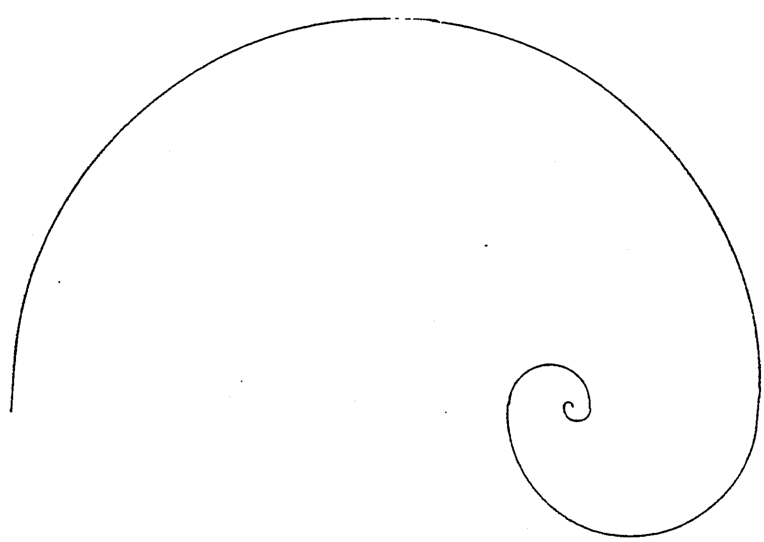
b) For $t = .5$, LIM is a $\sin(1/x)$ curve.

$D_5 =$



c) For $.5 < t \leq 1$, LIM is an arc.

For t=.9, $D_5 =$

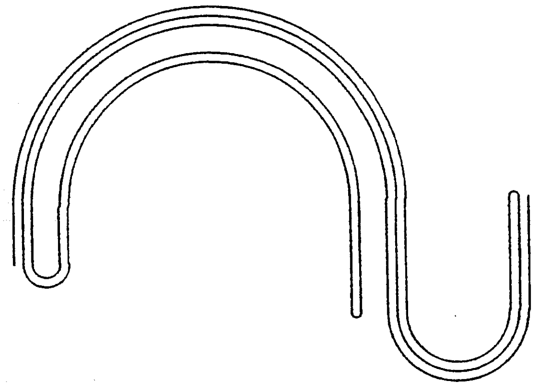


III. Dynamicist's function.

PL-map through $(0,t)$, $(.5,1)$, and $(1,0)$.

a) For $0 \leq t \leq .5$, LIM is indecomposable.

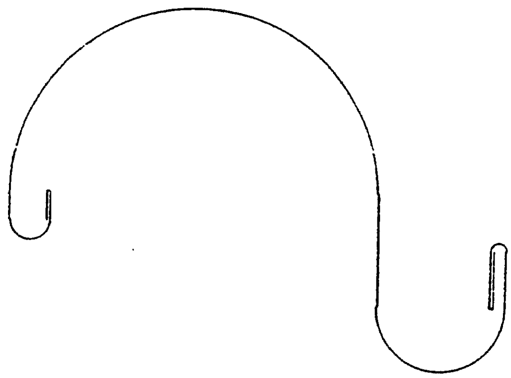
For $t=.5$, map is Golden Mean Map and LIM is a famous indecomposable continuum. For $t=.5$, $D_4 =$



b) For $.5 < t < .75$, LIM contains an indecomposable continuum.

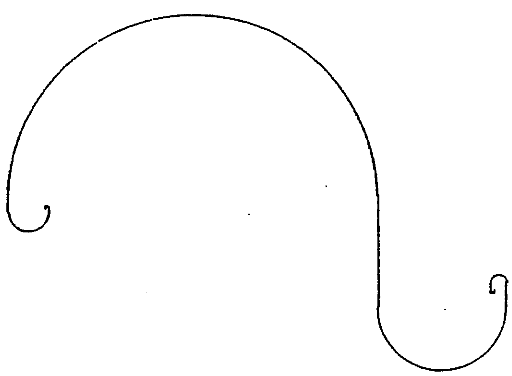
c) For $t=.75$, LIM is double-ended $\sin(1/x)$ curve.

$D_4 =$



d) For $.75 < t \leq 1$, LIM is an arc.

For $t=.9$, $D_5 =$



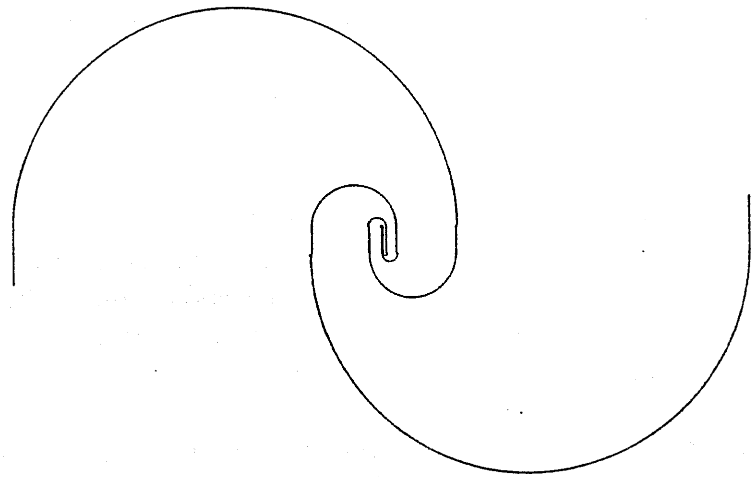
IV. For Bimodal Map.

For PL-map through (0,0), (1/3,r), (2/3, t), and (1,1) for $0 \le t < r \le 1$.

a) For $0 \le r \le t$, LIM is and arc.

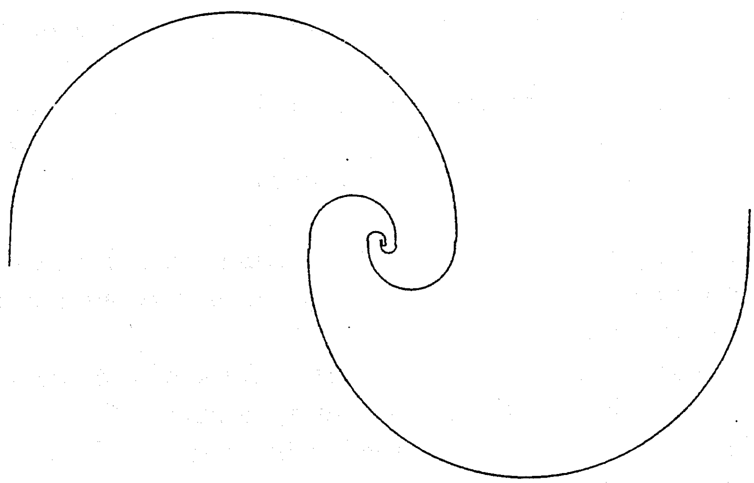
b) For $r = 2/3$, $t = 1/3$, LIM is a sin(1/x) with arcs limiting in from both sides.

$D_4 =$



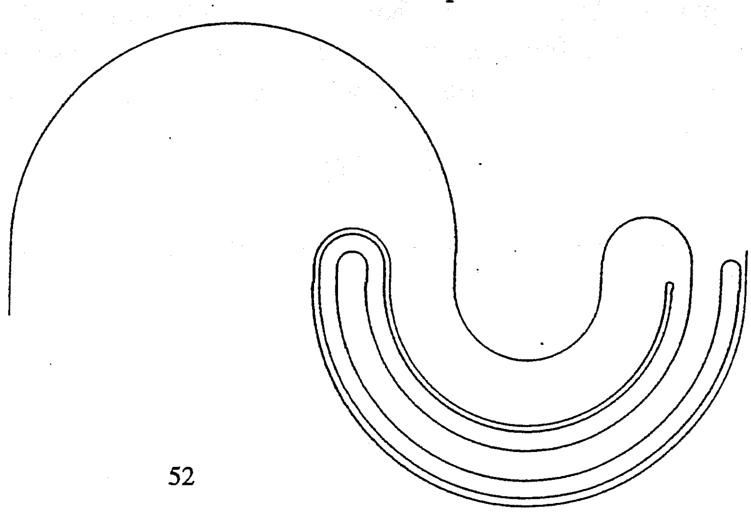
c) For $1/3 \le t < r < 2/3$ or $1/3 < t < r \le 2/3$, LIM is an arc.

For $t = 1/3$ and $r = .5$, $D_4 =$



d) For either $r > 2/3$ or $t < 1/3$, LIM contains an indecomposable continuum.

For $r = .9$, $t = .5$, $D_4 =$



In Codimension–k Manifold Decompositions the Discontinuity Set does not Locally Separate the Decompositon Space

FRANK H. SHAW
June 17, 1993

Given an $(n+k)$–manifold M decomposed upper semicontinuously into closed submanifolds of dimension n, what can be said of the structure of the decomposition space? How must the submanifolds be packed together? One might imagine that, if the codimension is small, rather stringent rules apply. What happens when the codimension is increased?

A *codimension–k manifold decomposition* is an upper semicontinuous decomposition of an $(n+k)$–manifold M without boundary into subcontinua having the shape of closed connected orientable n–manifolds. Throughout this paper we use the following notation: $\pi : M^{n+k} \to B$ is a codimension–k manifold decomposition of M, an $(n+k)$–manifold without boundary, with B denoting the decomposition space.

To date, much of the investigation of codimension–k manifold decompositions has taken the tack of assuming knowlege of the source manifold and the decomposition elements and then inferring information about the structure of the decomposition space.

In particular, much has come of restricting the codimension to $k \leq 3$. Here the decomposition elements have to fit together in nice ways and consequently, much can be said about B.

Fundamental to these studies is the concept of the *n-winding function* and its *continuity set*, which is the maximum open subset C of B over which the Leray cohomology sheaf in dimension n is locally constant. The n-winding function used in this paper is that used by Snyder [**Sn**], a variation of that used in [**CD**] and [**DW2**] in that it is defined on the n-th cohomology groups rather than the n-th homology groups. This is convenient because it gives an insight into the relationship between the winding function and the Leray cohomology sheaf.

For a fixed $b \in B$, let U_b be an open neighborhood of b such that there exists a (shape) retraction

$$r : \pi^{-1}(U_b) \to \pi^{-1}(b).$$

[0]This work was completed as a dissertation under the the supervision of John J. Walsh at the University of California, Riverside.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Fixing b, U_b, and r, we have, for each $y \in U_b$ the restriction homomorphism

$$h : H^n(\pi^{-1}(U_b); \mathbb{Z}) \to H^n(\pi^{-1}(y); \mathbb{Z}).$$

The composition of the inverse of the restriction isomorphism induced by r and h gives a well defined homomorphism

$$H^n(\pi^{-1}(b); \mathbb{Z}) \to H^n(\pi^{-1}(y); \mathbb{Z}) \tag{1}$$

between copies of $\mathbb{Z}$. The n-winding number of y with respect to b, denoted $\alpha_b(y)$, is the absolute value of the degree of this homomorphism.

We have:

Definition. The *continuity set* $C \subset B$ is the set

$$C = \{b \in B \mid \text{ there exists } U_b \text{ with } \alpha_b(b') = 1 \text{ for all } b' \in U_b\}.$$

Definition. The *discontinuity set* $D = B \setminus C$.

Definition. The *degeneracy set* of B, denoted K, is the set of all points $b \in D$ such that for each $W \subset B$ with $b \in W$, there exists $b' \in W$ with $\alpha_b(b') = 0$. Coram and Duvall [**CD**] proved that C is dense and open in B. It is in the discontinuity set, and, in particular, in the degeneracy set that the anomolies occur which make the study of codimension–k manifold decompositions interesting and difficult. Many of the results we have to date have been ferreted out by discoveries limiting the extent of the discontinuity and degeneracy sets. The paper in hand takes a general step in this direction, asserting that the discontinuity set does not locally separate B.

To evoke the proper context, we here restate the findings which suggest that the above result might be true.

In the codimension-1 case, we have that B is a one-manifold [**D1**], possibly with boundary but with $\partial B = \emptyset$ provided both M and the elements of G are orientable. Daverman proved that $C = \text{Int}(B)$. Thus, trivially, D does not locally separate B.

In the codimension-2 case [**DW1 DW2**] it has been shown that D is locally finite in B. This result was more than enough to prove that the decomposition space is a 2-manifold. Daverman was able to remove the orientable condition on M and showed that, for non-orientable M, B might possibly have boundary [**D2**]. Thus, in the codimension-2 case, D does not locally separate B.

Finally, Daverman [**D3**] found in his investigation of PL maps with manifold fibers that the discontinuity set is relegated to at most the $(k-2)$-skeleton of (the polyhedron) B:

THEOREM [DAVERMAN]. *Suppose $\pi : M \to B$ is simplicial. The continuity set C of $\pi : M \to B$ satisfies $B \setminus B^{k-2} \subset C$ and C meets each connected open subset of B in another connected set.*

The main result of the work in hand is provides a pleasing complement to Daverman's result.

THEOREM 1. *Let $\pi : M \to B$ be a codimension–k manifold decomposition of M, an (n+k)-manifold, into sets having the shape of closed oriented n–manifolds. Suppose that* $\dim(B) < \infty$. *Then D, the discontinuity set of π, measured with respect to $\mathbb{Z}$ coefficients, does not locally separate B.*

We present here a sketch of the proof of Theorem 1 dealing with the intuitively tractible special case of $\pi : S^4 \to B$ where $\pi^{-1}(b) \cong S^1 \forall b \in B$. It is still an open question whether or not such a decomposition can exist. If it does, Daverman **[D3]** has shown that it cannot be a piecewise linear map. In any case we sketch below the argument showing that the degeneracy set K of π in this case cannot locally separate B.

SKETCH OF PROOF: We assume that there exists a subset of K and a connected open $U \subset B$ such that $U \setminus K$ has multiple components U_i. For some U_i we further assume that $\partial U_i \setminus K \cong D^2$ and that there exists a subset $A \subset U \cup K$ such that

(1) $\partial A \subset K$
(2) $(A \setminus \partial A) \cap K = \emptyset$, and
(3) A does not separate B.

For a contradiction to the above we now look at $\pi^{-1}(A)$ in the source manifold S^4. We will argue that $\pi^{-1}(A)$ separates S^4 as is shown in Figure 1. Note that, since $(A \setminus \partial A) \subset C$, $\pi^{-1}(A \setminus \partial A) \cong D^2 \times S^1$. Furthermore, since $\partial A \subset K$, $\pi^{-1}(A)$ consists of $D^2 \times S^1$ together with $\pi^{-1}(\partial A) = S^1 \times S^1$ where the frontier of $D^2 \times S^1$ is wedged onto an $S^1 \subset \pi^{-1}(\partial A)$.

It is clear then that the generating element of $H^2(\pi^{-1}(\partial A)) \cong \mathbb{Z}$ is contained in $H^2(\pi^{-1}(A))$ and

$$H^2(\pi^{-1}(A)) \to H^2(\pi^{-1}(\partial A))$$

is a surjection. Following along the long exact sequence for $(\pi^{-1}(A), \pi^{-1}(\partial A))$ we have

$$H^3(\pi^{-1}(A), \pi^{-1}(\partial A)) \to H^3(\pi^{-1}(\partial A))$$

is an injection. Our simplifying assumptions give

$$H^3(\pi^{-1}(A), \pi^{-1}(\partial A)) \cong H^3(A \times S^1, \partial A \times S^1)$$

and the Kunneth formula shows that

$$H^3(A \times S^1, \partial A \times S^1) \cong H^2(A, \partial A) \otimes H^1(S^1) \cong \mathbb{Z}.$$

Thus $H^3(\pi^{-1}(A) \neq 0$ and by duality $H_1(S^4, S^4 \setminus \pi^{-1}(A)) \neq 0$. We see by

$$0 = H_1(S^4) \to H_1(S^4, S^4 \setminus \pi^{-1}(A)) \to H_0(S^4 \setminus \pi^{-1}(A)) \to H_0(S^4)$$

that

$$H_0(S^4 \setminus \pi^{-1}(A)) \not\cong H_0(S^4)$$

which implies that $\pi^{-1}(A)$ separates S^4. Since A does not separate B we have a contradiction. ▮

The generalization of the above sketch encounters numerous difficulties. The set A must be constructed carefully so that its complementary domains in B are controlled. The proof that $H^{n+k-1}(\pi^{-1}(A)) \neq 0$ involves a lengthy spectral sequence argument. Furthermore, since the theorem deals with the discontinuity set D instead of the degeneracy set K, it is necessary to manipulate the coefficient modules in order to bring the argument into line with the one sketched above.

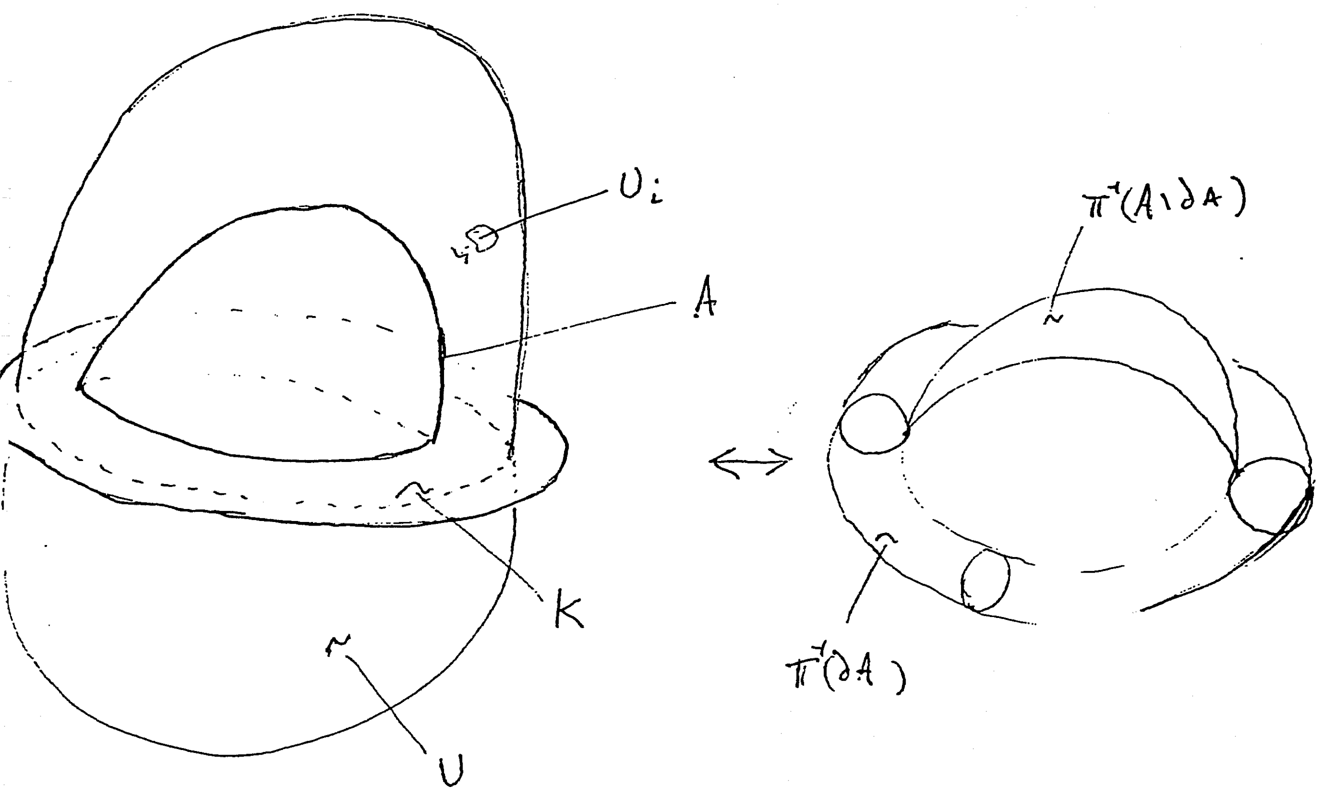


Figure 1

References

[CD] D. Coram and P. F. Duvall, *Finiteness theorems for approximate fibrations*, Trans. Amer. Math. Soc. **269** (1982), 383–394.

[D1] R. J. Daverman, *Decompositions of manifolds into codimension one manifolds*, Compositio Math. **55** (1985), 185–207.

[D2] ——————, *Decompositions into codimension two submanifolds: the nonorientable case*, Topology Appl. **24** (1986), 71–81.

[D3] ——————, *PL maps with manifold fibers*, To appear in, Jour. London Math. Soc..

[DW1] R. J. Daverman and J. J. Walsh, *Decompositions into codimension two spheres and approximate fibrations*, Topology Appl. **19** (1985), 103–121.

[DW2] ——————, *Decompositions of manifolds into codimension-2 submanifolds*, Trans. Amer. Math. Soc. **288** (1985), 273–291.

[Sn] D. F. Snyder, *Partially Acyclic Manifold Decompositions Yielding Generalized Manifolds*, Trans. Amer. Math. Soc. **325** (1991), 531–571.

IMA UMN

h-connected Groups and Spaces

Mathew Timm

Bradley University

1. Introduction and motivation.

It is clear that if p:X⟶Y is a (connected) finite sheeted cover of the simply connected manifold Y, then p is a homeomorphism from X to Y. Thus a connected simply connected manifold is one which has the property that its only finite sheeted covering spaces are copies of itself. Jungck[J] was investigating proper local homeomorphisms when he coined the term H-connected to describe this sort of situation.

Definition 1.1. Let Y be a first countable, connected, Hausdorff space. Then Y is *H-connected* iff whenever p:X⟶Y is a connected finite-sheeted covering space, it follows that p is a homeomorphism.

If one assumes that the above space Y is a compact manifold, there is the additional observation that for dim Y=1,2, Y is H-connected if and only if Y is simply connected. For dim Y=n $\geq$ 4, there are compact n-dimensional H-connected manifolds that are not simply connected. (See Timm[T1] for one such construction.) If dim Y=3 an interesting situation develops. Let $\mathfrak{RF}$ denote the class of all compact, connected, 3-manifolds, with residually finite π_1. Now note that if Y is finite dimensional, compact, and H-connected, then $H_1(Y)$ is trivial. Hence, by duality, a compact H-connected 3-manifold is a homology sphere. Now, if M $\in \mathfrak{RF}$ and is H-connected, one obtains that M is simply connected. So, for the class $\mathfrak{RF}$ one asks "Is every H-connected space in $\mathfrak{RF}$ a topological sphere?"

The following generalization of the idea of an H-connected space was suggested by the n-tori, $T^n = S^1 \times \ldots \times S^1$. Note that T^n has the property that if p:X⟶T^n is a connected finite sheeted cover then X is homeomorphic to T^n. However the given covering projection need not be a homeomorphism.

Definition 1.2. Let Y be a connected first countable Hausdorff space. Then Y is *h-connected* if and only if given any connected covering p:X⟶Y, it follows that X is homeomorphic to Y via some homeomorphism. A group G is *H-connected*, or *Hc*, if and only if given any monomorphism φ:H⟶G with $[G:\varphi(H)]<\infty$ it follows that $\varphi(H)=G$. Equivalently, G is Hc if and only if it has no nontrivial finite index subgroups. A group G is *h-connected,* or *hc*, if and only if whenever φ:H⟶G is a monomorphism it follows that $\varphi(H) \cong G$ via some isomorphism. Equivalently, G is hc every finite index subgroup of G is isomorphic to G. (To avoid confusion, read "h-connected" as "little h-connected" and "H-connected" as "big H-connected.") If G is h-connected, write G $\in \mathcal{hc}$ and if G is H-connected, write G $\in \mathcal{Hc}$. It is clear that $\mathcal{Hc} \subset \mathcal{hc}$.

The group theoretic notions are related to Hopfian conditions. In particular, if $G \in \mathcal{H}c$ and there is no monomorphism φ:G⟶G such that [G:φ(G)] is infinite, then G is cohophian. Daverman[D1], [D2] says that a group G is *hyperhopfian* if and only if for every homomorphism φ:G⟶G with φ(G) normal in G and G/φ(G) cyclic it follows that φ is an automorphism of G.

Fact 1.3. Assume G is finitely generated. Then, $G \in \mathcal{H}c$ if and only if $G \in hc$ and is hyperhopfian.

Proof. Observe that, if $G \in \mathcal{H}c$, there is no homomorphism φ:G⟶G such that φ(G) is normal and G/φ(G) infinite cyclic. Hence, if $G \in \mathcal{H}c$, then G is hyperhopfian. On the other hand suppose that $G \in hc$, G is finitely generated, and G is hyperhopfian. Assume that $G \notin \mathcal{H}c$. Then G has a subgroup H with $2 \leq [G{:}H] < +\infty$. By results in Section 2 we may, without loss of generality assume that H is normal and G/H is cyclic. Since $G \in hc$, there is an isomorphism φ:G⟶H. But G is hyperhopfian. Therefore, φ is an automorphism. Contradiction, since $\varphi(G) \neq G$.

As T^n shows, there are h-connected spaces that are not H-connected and, as its fundamental group $\mathbb{Z} \times \ldots \times \mathbb{Z}$ shows, there are hc groups that are not Hc. Furthermore, there are finitely presented, nonabelian hc groups that are not Hc. For example, let G be a finitely presented infinite simple group, e.g., see Higman[H], then $G \times \mathbb{Z}$ is a finitely presented, nonabelian hc group that is not Hc. If M is a 4-manifold with $\pi_1(M)=G$ then $M \times S^1$ is an h-connected 5-manifold with nonabelian π_1.

Section 2 of this manuscripts presents some immediate corollaries of these definitions and gives several conditions for an h-connected space to be H-connected. Section 3 looks at h-connected nilpotent groups.

2. Elementary properties.

In this section the word *space* will be used to denote a connected, first countable Hausdorff space that is locally path connected and semi-locally 1-connected. These properties assure that the spaces under consideration have a simply connected universal covering space and that there is a one-to-one correspondence between covering spaces of a given space and subgroups of its fundamental group. See Spanier[S]. Typically, one may assume they are compact. The first few results in this section are stated without proof. Their proofs follow easily from the definitions and the results preceding them.

Fact 2.1. Let $G \in \mathcal{H}c$ and let G' denote its commutator subgroup. If G is finitely generated, then G/G' is trivial. So if M is an H-connected manifold with finitely generated π_1, then $H_1(M)=0$.

Note that if G is not assumed to be finitely generated then it is not the case that G/G' is trivial. For example, the additive group of rationals, $\mathbb{Q}$, has no nontrivial subgroups of finite index. So $G \in \mathcal{H}c$. Since $\mathbb{Q}$ is abelian it has trivial commutator subgroup. So $\mathbb{Q}/\mathbb{Q}' \cong \mathbb{Q}$ which is not finitely generated.

Fact 2.2. Let M be a space with finitely generated fundamental group. Then M is H-connected if and only if $\pi_1(M) \in \mathcal{H}c$.

Observe that there are spaces Y with $\pi_1(Y) \in hc$ while Y is not h-connected. For example, $S^1 \vee I$, the join of the circle and the closed interval has hc fundamental group but is not itself h-connected. There is however at least one situation where it is easy to see that the analog of Fact 2.2 is true for h-connected spaces. The reader may find that Fact 2.4 is of use in its proof.

Fact 2.3. Let M be a compact two dimensional manifold. Then M is h-connected if and only if $\pi_1(M) \in hc$.

Fact 2.4. Let M be a compact, h-connected space. If M is not H-connected, that is, if M has a nontrivial finite sheeted cover, then the Euler characteristic, $\chi(M)$, is zero.

Theorem 2.5. Let $G \in hc$. If G is finitely generated and $G \notin \mathcal{H}c$, then G has a normal subgroup H such that G/H is finite cyclic. Note that necessarily $H \cong G$.

Proof. Since $G \in hc \backslash \mathcal{H}c$, G has a subgroup K such that $2 \leq [G{:}K] < +\infty$. So G finitley generated implies that G has a normal subgroup $N \leq K$ with $2 \leq [G{:}K] \leq [G{:}N] < +\infty$. Choose $g \in G$ such that its image $\overline{g} = gN \in G/N$ is nontrivial and consider the cyclic subgroup $<\overline{g}> \leq G/N$. Let $<\overline{g}>^*$ denote its pullback under the canonical projection $G \rightarrow G/N \rightarrow 1$. Then $<\overline{g}>^*$ is of finite index in G. So $G \in hc$ implies that there is an isomorphism $\varphi{:}G \rightarrow <\overline{g}>^*$. Take $H = \varphi^{-1}(N)$. Since N is normal in $<\overline{g}>^*$ it follows that H is normal in G. Furthermore $G/H \cong <\overline{g}>$ is finite cyclic.

Corollary 2.6. Let $G \in hc \backslash \mathcal{H}c$. Then G is neither hyperhopfian nor cohopfian.

Corollary 2.7. Let M be a space with finitely generated fundamental group. M is H-connected if and only if M is h-connected and $H_1(M)=0$.

Proof. If M is H-connected then it is trivially h-connected and by Fact 2.1, $H_1(M)=0$. So assume that M is h-connected and $H_1(M)=0$. Now assume that M is not H-connected. Then M has a nontrivial finite sheeted covering space. Hence $\pi_1(M) \in hc\backslash\mathcal{H}c$. So $\pi_1(M)$ has a normal subgroup N such that $\pi_1(M)/N$ is finite cyclic. In particular, $\pi_1(M)/N$ is a nontrivial abelian group. Thus N contains the commutator subgroup of $\pi_1(M)$ and it follows that $H_1(M) \neq 0$. Contradiction.

While the next characterization of an H-connected space clearly follows from the theorem, the question it answers originally arose after reading Hicks and Saligia's[HS] work on fixed points of non-self maps and a conversation with G. Jungck.

Corollary 2.8. Let M be a space. If M is h-connected and satisfies the fixed point property (FPP) for homeomorphisms, then M is H-connected.

Proof. Assume that M is h-connected and satisfies FPP for homeomorphisms. (A space X satisfies FPP for homeomorphisms if and only if every homeomorphism $h:X \to X$ has a point x such that $f(x)=x$.) Assume that M is not H-connected. Then by the theorem and the fact that M is a space, M has a regular nontrivial finite cyclic cover $p:\widetilde{M} \to M$ where $\widetilde{M}$ is homeomorphic to M. Let $\alpha \in \mathrm{Aut}_M \widetilde{M}$ be a generator of $\mathrm{Aut}_M \widetilde{M}$. Then, in particular, $\alpha:\widetilde{M} \to \widetilde{M}$ is a homeomorphism. Since M has FPP for homeomorphisms and FPP for local homeomorphisms is a topological property, $\widetilde{M}$ has FPP for local homeomorphisms. Therefore α has a fixed point, x_0. Thus for all $n \in \mathbb{Z}$, $\alpha^n(x_0)=x_0$. Hence, $p^{-1}(p(x_0))=\{x_0\}$. But p is a finite nontrivial cyclic cover of M and α is a generator of the nontrivial finite cyclic group $\mathrm{Aut}_M \widetilde{M}$. Therefore $\#(p^{-1}(p(x_0)))=|\alpha| \geq 2$. Contradiction. Hence M is H-connected.

The converse to Corollary 2.8. is clearly false. The 2-sphere, S^2 comes to mind. It is simply connected, therefore H- and h-connected, and yet the antipodal map is a homeomorphism that has no fixed point. One wonders what topological condition can replace "FPP for homeomorphisms" to obtain necessary and sufficient conditions for h-connectedness to imply H-connectedness. The next result suggests such a property.

Definition 2.9. Let Y be first countable, Hausdorff, and connected. (Note that we do not assume that Y is a *space.*) A closed subset K of Y is locally separating in Y if and only if $K \neq \emptyset$ and there is an open subset $V \subset Y$ with $K \subset V$ such that $V\backslash K=A \bigcup B$ with A and B nonempty disjoint open subsets such that $K \subset \bar{A} \cap \bar{B}$. We say that Y satisfies LG if and only if whenever K is a closed locally separating subset of Y it follows that K is globally separating.

Lemma 2.10. Let Y be first countable Hausdorff. (Again note that it is not assumed that Y is a *space.*) If Y is H-connected, then Y is h-connected and satisfies Property LG.

Proof. Since Y is H-connected it is trivially h-connected. For a proof that Y satisfies LG see Jungck[J, 7.6].

An obviously forced partial converse can be obtained by mimicking Theorem 2.5 and the proof of the "if" portion of Corollary 2.7 in the topological setting. The statement follows. Its proof is a straightforward consequence of Jungck[J, 7.7].

Lemma 2.11. Let Y be first countable, Hausdorff, and connected. Assume that Y satisfies $(*)$.

$(*)$ Y has a nontrivial finite sheeted covering space, M, that has a finite cyclic regular cover.

If Y is h-connected and satisfies LG, then Y is H-connected.

3. On the upper central series of hc groups

The results in this section are motivated by Conjecture 3.1. Complete proofs of these lemmas can be found in [T2]. A proof of a version of the conjecture appears at the end of the section. Good references for the group theory are Scott[Sc] and Hungerford[Hu].

Conjecture 3.1. Let $G \in hc$ be finitely generated. If G is nilpotent, then G is free abelian.

Corollary 3.2. Let M be a compact, sufficiently large, (PL) 3-manifold with $\pi_2(M)=0$. If $\pi_1(M) \in hc$ is nilpotent, and Conjecture 3.1 is true, then $\pi_1(M)$ is one of $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Proof. The indicated groups are the only free abelian ones that appear in Evans and Moser[EM].

Lemma 3.3. Let $G \in hc$ be finitely generated. If G is abelian, then G has no nontrivial torsion.

Proof (Sketch). This follows from the Fundamental Theorem of Finitely Generated Abelian Groups and the definition of an hc group.

Lemma 3.4. Suppose that G satisfies that maximal condition on abelian subgroups or equivalently satisfies ACC on abelian subgroups. (See [Sc, p.85]) If A is an abelian subgroup of G and φ:G$\rightarrow$G is a monomorphism such that $A \leq \varphi(A)$ then $A=\varphi(A)$.

Proof (Sketch). One uses the fact that $A \leq \varphi(A)$ to obtain ascending chain of abelian groups and then apply the fact that such a chain must terminate.

Lemma 3.5. Suppose $G \in hc$ and that G has a normal subgroup N such that for every monomorphism $\varphi: G \to G$ with $N \leq \varphi(G)$ and $[G:\varphi(G)] < \infty$ it follows that $\varphi(N)=N$. Then $G/N \in hc$.

Theorem 3.6. Let $G \in hc$ be a group that satisfies ACC on subgroups. Let

$$1=Z_0(G) \leq Z_1=Z_1(G) \leq Z_2(G) \leq \ldots. \leq Z_n(G) \leq \ldots..$$

be the upper central series for G. Then for all $n \in \mathbb{N}$, $G/Z_n(G) \in hc$.

Proof. The result clearly follows from the definition of Z_n, the fact that the class of groups satisfying ACC on subgroups is closed under quotients, and induction on n, provided that $G/Z_1(G) \in hc$.

Let $Z=Z_1(G)$. Assume that $[G/Z:H/Z]<\infty$. Since $[G/Z:H/Z]=[G:H]$ and $G \in hc$, it follows that there is an onto isomorphism $\varphi: G \to H$. Observe that since $Z \leq H \leq G$ it follows that $Z \leq Z_1(H)=\varphi(Z)$. So, by Lemma 3.4, $Z=Z_1(H)=\varphi(Z_1)$. Therefore, by Lemma 3.5, $G/Z \in hc$.

Theorem 3.7. Let $G \in hc$ be finitely generated. Assume that G is nilpotent of class n and

$$1=Z_0(G) \leq Z_1=Z_1(G) \leq Z_2(G) \leq \ldots. \leq Z_n(G)=G$$

is the upper central series of G. Then all of G, $G/Z_k(G)$ for $k=1,\ldots,n$, and $Z_{k+1}(G)/Z_k(G)$ are torsion free.

Proof. Observe that since G is finitely generated, nilpotent it satisfies ACC on subgroups by [Sc, 8.4.35] and [Sc, p. 85]. Hence the preceding results apply. Since G is nilpotent and satisfies ACC on subgroups it is supersolvable by [Sc, 7.2.7]. Therefore, by definition of supersolvable and M-group, G is an M-group. So, by [Sc, 7.1.11], G has a characteristic torsion free subgroup H of finite index. But $G \in hc$. So $[G:H]<\infty$ implies $G \cong H$. Therefore, G is torsion free. By induction, and Theorem 3.6, $G/Z_k(G)$ is torsion free for all $k=1,\ldots,n$. Since g is torsion free, it is clear that $Z_1(G)$ is torsion free. Hence, by [Sc, 6.4.26], it follows that $Z_{k+1}(G)/Z_k(G)$ is torsion free for all k.

Remark 3.8. If G is finitely generated it actually follows from the first part of the proof of Theorem 3.7 and [Sc, 7.2.21] that G is nilpotent if and only if G is supersolvable.

To prove a version of Conjecture 3.1, some notation is introduced. Say that $G \in \mathfrak{Z}_n$, if $G \in hc$ and is finitely generated, nilpotent with an upper central series in which $Z_n=G$. Note that $G \in \mathfrak{Z}_n$ does not mean that n is the least natural number for which $Z_n=G$. See [Sc, p.142]. Observe that every finitely generated $G \in hc$ that is nilpotent is in some, in fact, infinitely many $\mathfrak{Z}_n$.

Fact 3.9. If $\mathcal{Z}_2$ contains only finitely generated free abelian groups, then $\mathcal{Z}_n$ contains only finitely generated free abelian groups for all $n \geq 2$.

Proof. The proof is by induction on n. For n=2 the result follows by assumption. So suppose that for all $2 \leq k \leq n-1$, $\mathcal{Z}_k$ contains only free abelian groups. Suppose that $G \in \mathcal{Z}_n$. If G is abelian we are done. So, supposed that G is not abelian. Then G has an upper central series

$$1 \leq Z_1 = Z_1(G) \leq Z_2 \leq \ldots \leq Z_n = G.$$

Note that since G is non-abelian, $Z_1 \neq 1$ and $Z_1 \neq G$. By Theorem 3.6 we have that $G/Z_1 \in hc$. Furthermore, since G is finitely generated, G/Z_1 is finitely generated and since G is not abelian, $g(G/Z_1) = g \geq 1$. Finally, by observe that $G/Z_1 \in \mathcal{Z}_{n-1}$. Therefore, G/Z_1 is free abelian of rank $g \geq 1$. Therefore, $G \in \mathcal{Z}_2$. Hence G is finitely generated free abelian. Contradiction. Thus $\mathcal{Z}_n$ contains only finitely generated abelian groups. So the full result follows by induction.

references

[D1] Daverman, R.J., 3-manifolds with Geometric Structure and Approximate Fibrations, Indiana University Mathematics Journal, vol. 40, no. 4, 1452-1469, 1991.

[D2] Daverman, R.L., Hyperhopfian Groups and Approximate Fibrations, preprint.

[EM] Evans, B., and Moser, L., Solvable Fundamental Groups of Compact 3-manifolds, Trans. Amer. Math. Soc., vol.168, 189-210, 1972.

[HS] Hicks, T., and Saligia, L.M., Fixed points for Non-self Maps I, Int. J. Math. and M. Sc., to appear.

[H] Higman, G., Finitely Presented Infinite Simple Groups, Notes on Pure Math, no. 8, Australian National University-Canberra, 1974.

[Hu] Hungerford,T.W., Algebra, Springer-Verlag, 1974.

[J] Jungck, G.F., Local Homeomorphisms, Dissertationes Mathematicae, Polish Acad. Sci., 1983.

[Sc] Scott, W.R., Group Theory, Dover, 1987.

[S] Spanier, E.H., Algebraic Topology, McGraw-Hill, 1966.

[T1] Timm, M., A Compact H-connected Manifold that is not Simply Connected, Q&A, vol.10, no. 2, 159-164, 1992.

[T2] Timm, M., Finitely Generated Nilpotent hc Groups are Free Abelian, preprint.

SOME INTERESTING CONTRACTIBLE 3-MANIFOLDS

Fred Tinsley and David G. Wright

1. Introduction. The classical contractible 3-manifolds, Whitehead manifolds and related manifolds that are *nice at infinity* , are known to not cover any manifolds other than themselves by a trivial covering projection [My], [Wr]. By *nice at infinity* we mean that the 3-manifold is *eventually end irreducible* as described in [B-T] or [B]. Equivalently, we mean that the 3-manifold is *eventually* π_1*-injective at infinity* as described in [Wr].

Definition 1.1. A topological space X is said to be *eventually* π_1*-injective at infinity* if there is a fixed compact set K (a core) of X so that for every compact set A there is a compact set B so that loops in $X - B$ which are inessential in $X - K$ are also inessential in $X - A$.

Informally, we think of this property as stating that loops close to infinity which are inessential missing the core are inessential close to infinity. This condition is really a very mild condition which is satisfied by all the classical contractible manifolds including the genus one Whitehead manifolds for which any torus in a defining sequence serves as a core.

In this note we construct contractible manifolds that are not eventually π_1-injective at infinity and yet still have the property that they cover no manifold other than themselves.

2. Whitehead manifolds. A genus one Whitehead manifold [Wh] is a 3-manifold that is the monotone union of solid tori T_i so that $T_i \subset \operatorname{Int} T_{i+1}$, T_i is contractible in T_{i+1}, but T_i does not lie in a ball contained in T_{i+1}.

It is a well-known that T_i does not lie in a ball in the Whitehead manifold. McMillan has made an extensive study of genus one Whitehead manifolds [Mc]. He has shown the existence of uncountably many different such manifolds. The key ingredient in McMillan's proof is a lemma of Schubert [S]. We first state a definition and then Schubert's lemma.

Definition 2.1. If T_1 and T_2 are solid tori with $T_1 \subset T_2$, the *geometric index* of T_1 in T_2 is the minimal number of points of intersection of a centerline of T_1 with a meridional disk of T_2. See [Mc] or [S] for more details.

Lemma 2.2 (Schubert) Let $T_1 \subset T_2 \subset T_3$ be solid tori so that the geometric index of T_1 in T_2 is p and the geometric index of T_2 in T_3 is q. Then the geometric index of T_1 in T_3 is pq.

For each sequence $\alpha = (\alpha_i)$ of positive integers we construct a specific Whitehead manifold W_α so that $W_\alpha = \cup T_i$ so that T_i goes around T_{i+1} $2\alpha_i$ times geometrically and zero times algebraicly. McMillan [Mc] has shown that if α and β are sequences of distinct odd primes such that an infinite number of primes occur in α which do not occur in β, then W_α and W_β are topologically different.

3. Contractible 3-manifolds that do not cover. We now indicate how to construct the promised interesting contractible 3-manifold which we call W. The manifold W contains disjoint open sets V and U_i, $-\infty < i < \infty$. Each U_i is a Whitehead manifold corresponding to a sequence (α_{ij}) of distinct odd primes as described above. The set V is homeomorphic with Euclidean 3-space. The closures of the open sets U_i and V are manifolds with boundary $\overline{U_i}$ and $\overline{V}$, respectively. The boundary of $\overline{U_i}$ equals a plane P_i, the planes P_i are disjoint, and the boundary of $\overline{V}$ equals $\bigcup_{i=-\infty}^{\infty} P_i$. The manifold W is the union of the $\overline{U_i}$ and $\overline{V}$. Finally for some k the sequence (α_{kj}) of distinct odd primes contains and infinite number of primes that do not occur in (α_{ij}) for $i \neq k$.

The details of the construction and the proof that this cannot be a non-trivial covering space are given in [T-W]. However we now sketch the intuition.

Step 1. Any homeomorphism h of W gives rise to a permutation of the $\{U_i\}$. The open set U_i corresponds to $h \cdot U_i = U_j$ if for any compact set K in U_i, $h(K)$ can be isotoped into U_j. An argument similar to McMillan's using Schubert's Lemma shows that $h \cdot U_k = U_k$ for any homeomorphism of W.

Step 2. If h is a covering translation of W, then $h \cdot U_k = U_k$, and so it is "almost" true that $h(U_k) = U_k$. But by [My] or [Wr], U_k supports only the trivial covering translation so h itself must be trivial.

References

[B-T] M. G. Brin and T. L. Thickstun, *3-manifolds which are end 1-movable*, Memoirs Am. Math. Soc. **81** , American Mathematical Society (1989).

[B] E. M. Brown, *Contractible 3-manifolds of finite genus at infinity*, Trans. Amer. Math. Soc. **245** (1978), 503-514.

[Mc] D. R. McMillan, Jr., *Some contractible open 3-manifolds*, Trans. Amer. Math. Soc. **102** (1962), 373-382.

[S] H. Schubert, *Knoten und Vollringe*, Acta Math. **90** (1953), 132-286.

[T-W] F. Tinsley and D. G. Wright, Some contracitble manifolds and coverings of manifolds in dimension three, preprint.

[Wr] D. G. Wright, *Contractible open manifolds which are not covering spaces,* Topology **31** (1992), 281-291.

[Wh] J. H. C. Whitehead, *A certain open manifold whose group is unity*, Quart. J. Math. **6** (1935), 268-279.

Department of Mathematics
Collorado College
Colorado Springs, CO 80903

Department of Mathematics
Brigham Young University
Provo, UT 84602

HOMOTOPY GROUPS OF COMPLEMENTS OF TOPOLOGICAL KNOTS

Gerard A. Venema

1. Introduction

Throughout this talk we will assume that h is a topological embedding of the $(n-2)$-sphere into the n-sphere, $h : S^{n-2} \to S^n$. We use Σ to denote the image set $h(S^{n-2})$ and refer to Σ as a *topological knot*. We will study the homotopy groups of the knot complement $S^n - \Sigma$ and try to determine the largest possible dimension in which the first nonstandard homotopy group can occur. Specifically, we wish to determine the largest possible value of k such that $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $i < k$, but $\pi_k(S^n - \Sigma) \neq \pi_k(S^1)$.

Members of the audience are probably thinking that answers to questions like that are well known. But what is known generally applies only to embeddings that are fairly nice. (Here "fairly nice" means that the embedding is smooth, piecewise linear, or locally flat.) The purpose of this talk is to demonstrate that the answer is quite different in the topological setting.

2. Homotopy groups of complements of smooth knots

Let us begin by reviewing the well-known facts in the smooth setting. When combined, the two facts below indicate that the first nontrivial homotopy group of the complement of a smooth knot can occur in any dimension below the middle dimension, but cannot occur in the middle dimension or above. Similar facts hold in the PL and locally flat settings.

Fact 1. *For each k, $1 \leq k < n/2$, there is a smooth embedding $h : S^{n-2} \to S^n$ such that $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $1 \leq i < k$ but $\pi_k(S^n - \Sigma) \neq \pi_k(S^1)$.*

Examples can be constructed which are boundaries of manifolds obtained by attaching handles to the standard $(n+1, n-1)$-ball pair [3]. Thus the examples are actually slice knots.

Fact 2. *If h is smooth and $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $1 \leq i < n/2$, then $S^n - \Sigma$ has the homotopy type of S^1.*

Fact 2 is proved by showing that the homology of the universal cover of $S^n - \Sigma$ vanishes. If n is odd, this follows quite easily from Poincaré duality. If n is even, more care is needed to prove that the homology group in the middle dimension vanishes. One approach is to use "Milnor Duality" [2] .

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Milnor Duality. Under certain conditions, an infinite cyclic cover of a compact n-manifold will have the duality properties of an $(n-1)$-manifold. In other words,

$$H_k(\widetilde{M}^n; F) \cong H^{n-k-1}(\widetilde{M}^n, \partial\widetilde{M}^n; F)$$

for any field F.

Milnor Duality can be used to prove Fact 2 in case n is even. In that case the homology in dimension $n/2$ is dual to the cohomology in dimension $(n/2-1)$ and is, therefore, trivial.

3. Homotopy groups of complements of topological knots

In the case of topological knots, the first nontrivial homotopy group can occur above the middle dimension. The surprising fact is that it can occur in any dimension up to $n-2$. The following example will appear in [4].

Example. *For each n and k, $1 \leq k \leq n-2$, there exists a topological embedding $h : S^{n-2} \to S^n$ such that $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $i < k$ but $\pi_k(S^n - \Sigma) \neq 0$. The embedding is smooth except at one point.*

If $k \geq n/2$, the knot in the example must necessarily be wild (because of Fact 2). There is a close relationship between the global homotopy groups of the knot complement and the homotopy groups of the end of the complement. We explore this relationship in the next three results (to appear in [4]).

Notation. We use W to denote the knot complement $S^n - \Sigma$, $\widetilde{W}$ to denote the infinite cyclic cover of W, and ϵ to denote the end of W.

Theorem 1. *If $\pi_i(\epsilon) \cong \pi_i(S^1)$ for $i \leq n-k-1$, then $H_k(\widetilde{W}; F) \cong H^{n-k-1}(\widetilde{W}; F)$ for every field F.*

Corollary 1. *If $\pi_i(W) \cong \pi_i(\epsilon) \cong \pi_i(S^1)$ for $i \leq k$, then $H_i(\widetilde{W}; \mathbb{Z}) = 0$ for $i \geq n-k-1$.*

Corollary 2. *If $\pi_i(W) \cong \pi_i(\epsilon) \cong \pi_i(S^1)$ for $i < n/2$, then W has the homotopy type of S^1.*

Conversely, the homotopy groups of the end of W can be controlled by controlling the global homotopy groups of W. In particular, if $\pi_i(W) \cong \pi_i(S^1)$ for every i, then any nontrivial homotopy group of the end *must* appear in dimension 1.

A geometric version of the following result was proved by Hollingsworth and Rushing [1]. We give an elementary proof based on duality.

Theorem 2. *If $\pi_i(W) \cong \pi_i(S^1)$ for every i and $\pi_1(\epsilon) \cong \mathbb{Z}$, then $\pi_i(\epsilon) = 0$ for $2 \leq i \leq n-3$ and $\pi_{n-2}(\epsilon) \cong \mathbb{Z}$.*

If we combine Theorem 2 with Corollary 2 we get the following result. It is a topological version of Fact 2.

Corollary 3. *If $\pi_i(W) \cong \pi_i(\epsilon) \cong \pi_i(S^1)$ for $i < \frac{n}{2}$, then W has the proper homotopy type of $S^1 \times \mathbb{R}^{n-1}$.*

4. A NONCOMPACT VERSION OF MILNOR DUALITY

The proof of Theorem 1 is based on a noncompact version of Milnor Duality. We assume the following notation for the remainder of this section: W is a noncompact PL n-manifold, W has one end ϵ, and $p : \widetilde{W} \to W$ denotes an infinite cyclic cover. As in Milnor's original work, it is convenient to use coefficients in a field F.

Definition.

$$H^k(\widetilde{W}, \tilde{\epsilon}; F) = \varinjlim H^k(\widetilde{W}, p^{-1}(U); F)$$

where the limit is taken over the collection of all neighborhoods U of the end ϵ, ordered by inclusion.

Definition. We will say that $H_k(\widetilde{W}, \tilde{\epsilon}; F)$ *is profinitely generated over* F if for every neighborhood U of ϵ there exists a neighborhood V of ϵ, $V \subset U$, such that the image of $H_k(\widetilde{W}, p^{-1}(V); F)$ in $H_k(\widetilde{W}, p^{-1}(U); F)$ is finitely generated over F.

In [4] we prove the following noncompact version of Milnor Duality and use it to prove Theorem 1, above.

Theorem 3. *If $H_i(\widetilde{W}, \tilde{\epsilon}; F)$ is profinitely generated over F for $n-k-2 \le i \le n-k$, then $H_k(\widetilde{W}; F) \cong H^{n-k-1}(\widetilde{W}, \tilde{\epsilon}; F)$.*

REFERENCES

[1] J. G. Hollingsworth and T. B. Rushing, *Homotopy characterization of weakly flat codimension 2 spheres*, Amer. J. Math. **98** (1976), 385–394.

[2] J. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds, Prindle, Weber and Schmidt, Boston, 1968, pp. 115–133.

[3] D. W. Sumners, *Homotopy torsion in codimension two knots*, Proc. Amer. Math. Soc. **24** (1970), 229–240.

[4] G. A. Venema, *Duality on noncompact manifolds and complements of topological knots*, Proc. Amer. Math. Soc. (to appear).

CALVIN COLLEGE, GRAND RAPIDS, MICHIGAN 49546 USA

E-mail address: venema@calvin.edu