Universal torsion, $L^2$-invariants, polytopes and the Thurston norm

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Given a group $G$, its group von Neumann algebra is defined to be

$$\mathcal{N}(G) = B(L^2(G), L^2(G))^G = \mathbb{C}G^{\text{weak}}.$$ 

It comes with a natural trace

$$\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$ 

For $x = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we have $\text{tr}_{\mathcal{N}(G)}(x) = \lambda_e$. 

If $G$ is finite, $\mathcal{N}(G) = \mathbb{C}G$. 

If $G = \mathbb{Z}$, we get identifications

$$\mathcal{N}(\mathbb{Z}) = L^\infty(S^1);$$

$$\text{tr}_{\mathcal{N}(\mathbb{Z})}(f) = \int_{S^1} f(z) d\mu.$$
Review of classical $L^2$-invariants

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The trace yields via the Hattori-Stallings rank the dimension function with values in \([0, \infty)\) for finitely generated projective \(\mathcal{N}(G)\)-modules due to Murray-von Neumann.

This has been extended to the dimension function \(\dim_{\mathcal{N}(G)}\) with values in \([0, \infty]\) for arbitrary \(\mathcal{N}(G)\)-modules, which has still nice properties such as additivity and cofinality, by Lück.

**Definition (\(L^2\)-Betti number)**

Let \(Y\) be a \(G\)-space. Define its \(n\)th \(L^2\)-Betti number

\[
b_n^{(2)}(Y; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C^*_\text{sing}(Y))) \in [0, \infty].
\]

Sometimes we omit \(\mathcal{N}(G)\) from the notation when \(G\) is obvious.
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\beta^{(2)}_n(Y; N(G)) := \dim_{N(G)}(H_n(N(G) \otimes_{ZG} C^\text{sing}_*(Y))) \in [0, \infty].
\]

Sometimes we omit \(N(G)\) from the notation when \(G\) is obvious.
This definition is the end of a long chain of generalizations of the original notion due to Atiyah which was motivated by index theory. He defined for a $G$-covering $\overline{M} \to M$ of a closed Riemannian manifold

$$b_n^{(2)}(\overline{M}) := \lim_{t \to \infty} \int_{\mathcal{F}} \text{tr} \left( e^{-t \cdot \Delta_n(x, \overline{x})} \right) d\text{vol}_{\overline{M}}.$$

- If $G$ is finite, we have
  $$b_n^{(2)}(\overline{X}) = \frac{1}{|G|} \cdot b_n(X).$$
- If $G = \mathbb{Z}$, we have
  $$b_n^{(2)}(\overline{X}) = \dim_{\mathbb{C}[\mathbb{Z}]^{(0)}} (\mathbb{C}[\mathbb{Z}]^{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} H_n(\overline{X}; \mathbb{C})) \in \mathbb{Z}.$$
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In the sequel 3-manifold means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

**Theorem (Lott-Lück)**

For every 3-manifold $M$ all $L^2$-Betti numbers $b_n^{(2)}(\tilde{M})$ vanish.

We are interested in case where all $L^2$-Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called $L^2$-torsion.
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\( L^2 \)-torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing the notion of analytic Ray-Singer torsion. It can also be defined in terms of the cellular \( \mathbb{Z}G \)-chain complex, generalizing of the Reidemeister torsion.

The definition of \( L^2 \)-torsion is based on the notion of the Fuglede-Kadison determinant which is a generalization of the classical determinant to the infinite-dimensional setting. It is defined for an element \( f \in \mathcal{N}(G) \) to be the non-negative real number

\[
\det^{(2)}(f) = \frac{1}{2} \cdot \int \ln(\lambda) \, d\nu_{f^*f} \in \mathbb{R}
\]

where \( \nu_{f^*f} \) is the spectral measure of the positive operator \( f^*f \).

If \( G \) is finite, then \( \det^{(2)}(f) = |G|^{-1} \cdot \ln(|\det(f)|) \).
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Theorem (Lück-Schick)

Let $M$ be a 3-manifold. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then

$$\rho^{(2)}(\tilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \text{vol}(M_i).$$
Definition \((K^w_1(\mathbb{Z}G))\)

Let \(K^w_1(\mathbb{Z}G)\) be the abelian group given by:

- **generators**
  
  If \(f : \mathbb{Z}G^m \to \mathbb{Z}G^m\) is a \(\mathbb{Z}G\)-map such that the induced \(N(G)\)-map \(N(G)^m \to N(G)^m\) map is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator \([f]\) in \(K^w_1(\mathbb{Z}G)\).

- **relations**

\[
\begin{bmatrix}
  f_1 & * \\
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\end{bmatrix} = [f_1] + [f_2];
\]

\[
[g \circ f] = [f] + [g].
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Define \(Wh^w(G) := K^w_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}\).
Universal $L^2$-torsion

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Let $G \to \bar{X} \to X$ be a $G$-covering of a finite $CW$-complex. Suppose that $\bar{X}$ is $L^2$-acyclic, i.e., $b_n^{(2)}(\bar{X})$ vanishes for all $n \in \mathbb{Z}$.

Then its universal $L^2$-torsion is defined as an element

$$\rho_u^{(2)}(\bar{X}) = \rho_u^{(2)}(\bar{X}; N(G)) \in K_1^w(\mathbb{Z}G).$$

- We do not present the actual definition but it is similar to the definition of Whitehead torsion in terms of chain contractions, one has now to use weak chain contractions.
- It has indeed a universal property on the level of weakly acyclic finite based free $\mathbb{Z}G$-chain complexes.
- By stating its main properties and its relation to classical invariant we want to convince the audience that this is a very interesting and powerful invariant which is worthwhile to be explored further.
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Let $G \to \tilde{X} \to X$ be a $G$-covering of a finite $CW$-complex. Suppose that $\tilde{X}$ is $L^2$-acyclic, i.e., $b_n^{(2)}(\tilde{X})$ vanishes for all $n \in \mathbb{Z}$.

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Homotopy invariance

Consider a pullback of $G$-coverings of finite $CW$-complexes with $f$ a homotopy equivalence

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
\bar{X} \xrightarrow{\bar{f}} \bar{Y}
\end{array}
\]

Suppose that $\bar{Y}$ is $L^2$-acyclic.
Then $\bar{X}$ is $L^2$-acyclic and the obvious homomorphism

\[\text{Wh}(G) \to \text{Wh}^w(G)\]

sends the Whitehead torsion $\tau(f : \bar{X} \to \bar{Y})$ to the difference of universal $L^2$-torsions $\rho_u^{(2)}(\bar{Y}) - \rho_u^{(2)}(\bar{X})$. 

Sum formula

Let $G \to \overline{X} \to X$ be a $G$-covering of a finite $CW$-complex $X$. Let $X_i \subseteq X$ for $i = 0, 1, 2$ $CW$-subcomplexes with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Let $\overline{X}_i \to X_i$ be given by restriction. Suppose that $\overline{X}_i$ is $L^2$-acyclic for $i = 0, 1, 2$.

Then $\overline{X}$ is $L^2$-acyclic and we get

$$\rho_u^{(2)}(\overline{X}; \mathcal{N}(G)) = \rho_u^{(2)}(\overline{X}_1; \mathcal{N}(G)) + \rho_u^{(2)}(\overline{X}_2; \mathcal{N}(G)) - \rho_u^{(2)}(\overline{X}_0; \mathcal{N}(G)).$$

There are product formulas or more general formulas for fibrations and for proper $S^1$-actions which we will not state.

Whenever an invariant comes from the universal torsion by applying a group homomorphism, these formulas automatically extend to the other invariant.
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Whenever an invariant comes from the universal torsion by applying a group homomorphism, these formulas automatically extend to the other invariant.
Example (Trivial $G$)

- Suppose that $G$ is trivial. Then we get isomorphisms

$$\text{K}_1^w(\{1\}) \cong \text{K}_1(\mathbb{Q}) \xrightarrow{\text{det}} \mathbb{Q}^×$$

$$\text{Wh}_1^w(\{1\}) \cong \{ r \in \mathbb{Q} \mid r > 0 \}.$$

- A finite based free $\mathbb{Z}$-chain complex $C_*$ is $L^2$-acyclic if and only if each of its homology groups is finite. Under the isomorphism above the universal torsion of $C_*$ is sent to $\prod_{n \geq 0} |H_n(C_*)|^{(-1)^n}$.

- For finite $CW$-complexes we essentially have the same information as the Milnor torsion.

- If $G$ is finite, we rediscover essentially the classical Reidemeister torsion.
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The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

\[ \mathbb{Z}G \longrightarrow \mathcal{N}(G) \]
\[ \downarrow \quad \downarrow \]
\[ \mathcal{D}(G) \longrightarrow \mathcal{U}(G) \]

- \( \mathcal{U}(G) \) is the algebra of affiliated operators. Algebraically it is just the Ore localization of \( \mathcal{N}(G) \) with respect to the multiplicatively closed subset of non-zero divisors.

- \( \mathcal{D}(G) \) is the division closure of \( \mathbb{Z}G \) in \( \mathcal{U}(G) \), i.e., the smallest subring of \( \mathcal{U}(G) \) containing \( \mathbb{Z}G \) such that every element in \( \mathcal{D}(G) \), which is a unit in \( \mathcal{U}(G) \), is already a unit in \( \mathcal{D}(G) \) itself.
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• If $G$ is finite, it is given by

$$
\begin{align*}
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\mathbb{Q}G & \longrightarrow \mathbb{C}G
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$$

• If $G = \mathbb{Z}$, it is given by

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\mathbb{Z}[\mathbb{Z}] & \longrightarrow L^\infty(S^1) \\
\mathbb{Q}[\mathbb{Z}]^{(0)} & \longrightarrow L(S^1)
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If $G$ is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.
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Conjecture (Atiyah Conjecture for torsionfree groups)

Let $G$ be a torsionfree group. It satisfies the Atiyah Conjecture if $D(G)$ is a skew-field.

- A torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension
  \[
  \dim_{\mathcal{N}(G)} \left( \ker \left( r_A : \mathcal{N}(G)^m \to \mathcal{N}(G)^n \right) \right)
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  is an integer. In this case this dimension agrees with
  \[
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- The Atiyah Conjecture implies the Kaplansky Conjecture saying that for any torsionfree group and field of characteristic zero $F$ the group ring $FG$ has no non-trivial zero-divisors.
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A torsionfree $G$ satisfies the Atiyah Conjecture if and only if each of its finitely generated subgroups does.

Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any $G$-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension $d$ we have $b_n^{(2)}(\overline{M};) \in \mathbb{Z}$ for every $n \geq 0$.

There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

However, there exist closed Riemannian manifolds whose universal coverings have an $L^2$-Betti number which is irrational, see Austin, Grabowski.
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Theorem (Linnell, Schick)

1. Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group $G$ which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.

2. If $G$ is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.

This Theorem and results by Waldhausen show for the fundamental group $\pi$ of a 3-manifold that it satisfies the Atiyah Conjecture and that $\text{Wh}(\pi)$ vanishes.
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If $G$ belongs to $C$, then the natural map

$$K^w_1(\mathbb{Z}G) \xrightarrow{\cong} K_1(D(G))$$

is an isomorphism.

- Its proof is based on identifying $D(G)$ as an appropriate Cohen localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic $K$-theory.
- There is a Dieudonné determinant which induces an isomorphism

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In particular we get for $G = \mathbb{Z}$

$$K_1^w(D(\mathbb{Z})) \cong \mathbb{Q}[\mathbb{Z}]^{(0)} \setminus \{0\}.$$ 

It turns out that then the universal torsion is the same as the Alexander polynomial of an infinite cyclic covering, as it occurs for instance in knot theory.
Twisting $L^2$-invariants

- Consider a CW-complex $X$ with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X; \mathbb{Z}) = \text{hom}(\pi; \mathbb{Z})$.
- For $t \in (0, \infty)$, let $\phi^* C_t$ be the 1-dimensional $\pi$-representation given by
  
  $$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}. $$

- One can twist the $L^2$-chain complex of $X$ with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion

  $$\mathbb{C}G \to \mathbb{C}G, \quad \sum g \cdot \lambda_g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g. $$

- Notice that for irrational $t$ the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.
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Thus we obtain the $\phi$-twisted $L^2$-torsion function

$$\rho(\tilde{X}; \phi): (0, \infty) \to \mathbb{R}$$

sending $t$ to the $\mathbb{C}_t$-twisted $L^2$-torsion.

Its value at $t = 1$ is just the $L^2$-torsion.

On the analytic side this corresponds for closed Riemannian manifold $M$ to twisting with the flat line bundle $\tilde{M} \times_\pi \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the $\pi$-action on $\mathbb{C}_t$ is not isometric.
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Theorem (Lück)

Suppose that \( \tilde{X} \) is \( L^2 \)-acyclic.

1. The \( L^2 \) torsion function \( \rho^{(2)} := \rho^{(2)}(\tilde{X}; \phi) : (0, \infty) \to \mathbb{R} \) is well-defined.

2. The limits \( \lim_{t \to \infty} \frac{\rho^{(2)}(t)}{\ln(t)} \) and \( \lim_{t \to 0} \frac{\rho^{(2)}(t)}{\ln(t)} \) exist and we can define the degree of \( \phi \)

\[ \deg(X; \phi) \in \mathbb{R} \]

to be their difference.

3. There is a \( \phi \)-twisted Fuglede-Kadison determinant

\[ \det_{tw, \phi}^{(2)} : K_1^w(\mathbb{Z}G) \to \text{map}((0, \infty), \mathbb{R}) \]

which sends \( \rho_{u}^{(2)}(\tilde{X}) \) to \( \rho^{(2)}(\tilde{X}; \phi) \).
**Definition (Thurston norm)**

Let $M$ be a 3-manifold and $\phi \in H^1(M; \mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_i(F) = \sum_{C \in \pi_0(M)} \min\{-\chi(C), 0\}.$$ 

Thurston showed that this definition extends to the real vector space $H^1(M; \mathbb{R})$ and defines a seminorm on it.

If $F \to M \xrightarrow{p} S^1$ is a fiber bundle and $\phi = \pi_1(p)$, then

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Theorem (Friedl-Lück)

Let $M$ be a 3-manifold. Then for every $\phi \in H^1(M; \mathbb{Z})$ we get the equality

$$\deg(M; \phi) = x_M(\phi).$$
Polytope homomorphism

Consider the projection

$$\text{pr}: G \rightarrow H_1(G)_f := H_1(G)/\text{tors}(H_1(G)).$$

Let $K$ be its kernel.

After a choice of a set-theoretic section of $\text{pr}$ we get isomorphisms

$$\mathbb{Z}K \ast H_1(G)_f \xrightarrow{\text{ir}} \mathbb{Z}G;$$

$$S^{-1}(\mathcal{D}(K) \ast H_1(G)_f) \xrightarrow{\text{ir}} \mathcal{D}(G),$$

where here and in the sequel $S^{-1}$ denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.
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Given \( x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) \ast H_1(G)_f \), define its support

\[
\text{supp}(x) := \{ h \in H_1(G)_f \mid h \in H_1(G)_f, u_h \neq 0 \}.
\]

The convex hull of \( \text{supp}(x) \) defines a polytope

\[
P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H^1(M; \mathbb{R}).
\]

This is the non-commutative analogon of the Newton polygon associated to a polynomial in several variables.

Recall the Minkowski sum of two polytopes \( P \) and \( Q \)

\[
P + Q = \{ p + q \mid p \in P, q \in Q \}
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It satisfies \( P_0 + Q = P_1 + Q \implies P_0 = P_1 \).
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Definition (Polytope group)

Let $\mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$ be the Grothendieck group of the abelian monoid of integral polytopes under the Minkowski sum modulo translations by elements in $H_1(G)_f$, where integral means that all extreme points lie on the lattice $H_1(G)_f$.

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) \ast H_1(G)_f)$.
- Hence we can define a homomorphism of abelian groups

$$P' : \left( S^{-1}(\mathcal{D}(K) \ast H_1(G)_f) \right)^{\times} \to \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$. 
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The composite

\[ K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(D(G)) \xrightarrow{\cong} D(G) \times \xrightarrow{\cong} \left( S^{-1}(D(K) \ast H_1(G)_f) \right) \times \xrightarrow{P'} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f) \]

factories to the polytope homomorphism

\[ P: Wh^w(G) \rightarrow \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f). \]
**Definition (Thurston polytope)**

Let $M$ be a 3-manifold. Define the **Thurston polytope** to be subset of $H^1(M; \mathbb{R})$

\[ T(M) := \{ \phi \in H^1(M; \mathbb{R}) \mid x_M(\phi) \leq 1 \}. \]

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**Theorem (Friedl-Lück)**

Let $M$ be a 3-manifold. Then the image of the universal $L^2$-torsion $\rho^{(2)}_u(\tilde{M})$ under the polytope homomorphism

\[ P : \text{Wh}^w(\pi_1(M)) \to \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f) \]

is represented by the dual of the Thurston polytope, which is an integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f = H_1(M; \mathbb{R}) = H^1(M; \mathbb{R})^*$. 
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We can assign to a finite $CW$-complex $X$ its universal $L^2$-torsion 

$$\rho^{(2)}(\tilde{X}) \in Wh^w(\pi),$$

provided that $\tilde{X}$ is $L^2$-acyclic and $\pi$ satisfies the Atiyah Conjecture.

These assumptions are always satisfied for 3-manifolds.

The Alexander polynomial is a special case.

One can read off from the universal $L^2$-torsion a polytope which for a 3-manifold is the dual of the Thurston polytope.

One can twist the $L^2$-torsion by a cycle $\phi \in H^1(M)$ and obtain a $L^2$-torsion function from which one can read off the Thurston norm.
**L^2-Euler characteristic**

**Definition (L^2-Euler characteristic)**

Let $Y$ be a $G$-space. Suppose that

$$h^{(2)}(Y; \mathcal{N}(G)) := \sum_{n \geq 0} b_n^{(2)}(Y; \mathcal{N}(G)) < \infty.$$  

Then we define its $L^2$-Euler characteristic

$$\chi^{(2)}(Y; \mathcal{N}(G)) := \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(Y; \mathcal{N}(G)) \in \mathbb{R}.$$
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Definition (*φ*-\(L^2\)-Euler characteristic)

Let \(X\) be a connected \(CW\)-complex. Suppose that \(\tilde{X}\) is \(L^2\)-acyclic. Consider an epimorphism \(\phi: \pi = \pi_1(M) \to \mathbb{Z}\). Let \(K\) be its kernel. Suppose that \(G\) is torsionfree and satisfies the Atiyah Conjecture.

Define the \(\phi\)-\(L^2\)-Euler characteristic

\[
\chi^{(2)}(\tilde{X}; \phi) := \chi^{(2)}(\tilde{X}; N(K)) \in \mathbb{R}.
\]

- Notice that \(\tilde{X}/K\) is not a finite \(CW\)-complex. Hence it is not obvious but true that \(h^{(2)}(\tilde{X}; N(K)) < \infty\) and \(\chi^{(2)}(\tilde{X}; \phi)\) is a well-defined real number.

- The \(\phi\)-\(L^2\)-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
Definition \((\phi-L^2\text{-Euler characteristic})\)

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Definition ($\phi$-$L^2$-Euler characteristic)

Let $X$ be a connected $CW$-complex. Suppose that $\tilde{X}$ is $L^2$-acyclic. Consider an epimorphism $\phi: \pi = \pi_1(M) \to \mathbb{Z}$. Let $K$ be its kernel. Suppose that $G$ is torsionfree and satisfies the Atiyah Conjecture.

Define the $\phi$-$L^2$-Euler characteristic

$$\chi^{(2)}(\tilde{X}; \phi) := \chi^{(2)}(\tilde{X}; N(K)) \in \mathbb{R}.$$ 

Notice that $\tilde{X}/K$ is not a finite $CW$-complex. Hence it is not obvious but true that $h^{(2)}(\tilde{X}; N(K)) < \infty$ and $\chi^{(2)}(\tilde{X}; \phi)$ is a well-defined real number.

The $\phi$-$L^2$-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
Let $f : X \to X$ be a selfhomotopy equivalence of a connected finite $CW$-complex. Let $T_f$ be its mapping torus. The projection $T_f \to S^1$ induces an epimorphism $\phi : \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

Then $\tilde{T}_f$ is $L^2$-acyclic and we get

$$\chi^{(2)}(\tilde{T}_f; \phi) = \chi(X).$$

**Theorem (Friedl-Lück)**

Let $M$ be a 3-manifold and $\phi : \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\tilde{M}; \phi) = \chi_M(\phi).$$
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**Theorem (Friedl-Lück)**

Let $M$ be a 3-manifold and $\phi : \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\tilde{M}; \phi) = x_M(\phi).$$
Higher order Alexander polynomials were introduced for a covering $G \to \overline{M} \to M$ of a 3-manifold by Harvey and Cochrane, provided that $G$ occurs in the rational derived series of $\pi_1(M)$.

At least the degree of these polynomials is a well-defined invariant of $M$ and $G$.

We can identify the degree with the $\phi$-$L^2$-Euler characteristic.

Thus we can extend this notion of degree also to the universal covering of $M$ and can prove the conjecture that the degree coincides with the Thurston norm.
Group automorphisms

Theorem (Lück)

Let \( f : X \to X \) be a self homotopy equivalence of a finite connected CW-complex. Let \( T_f \) be its mapping torus.

Then all \( L^2 \)-Betti numbers \( b_n^{(2)}(\tilde{T}_f) \) vanish.

Definition (Universal torsion for group automorphisms)

Let \( f : G \to G \) be a group automorphism of the group \( G \). Suppose that there is a finite model for \( BG \) and \( G \) satisfies the Atiyah Conjecture. Then we can define the universal \( L^2 \)-torsion of \( f \) by

\[
\rho_u^{(2)}(f) := \rho^{(2)}(\tilde{T}_f; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in Wh^w(G \rtimes_f \mathbb{Z})
\]

This seems to be a very powerful invariant which needs to be investigated further.
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This seems to be a very powerful invariant which needs to be investigated further.
It has nice properties, e.g., it depends only on the conjugacy class of $f$, satisfies a sum formula and a formula for exact sequences.

If $G$ is amenable, it vanishes.

If $G$ is the fundamental group of a compact surface $F$ and $f$ comes from an automorphism $a: F \to F$, then $T_f$ is a 3-manifold and a lot of the material above applies.

For instance, if $a$ is irreducible, $\rho_u^{(2)}(f)$ detects whether $a$ is pseudo-Anosov since we can read off the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition of $T_f$. 

Suppose that $H_1(f) = \text{id}$. Then there is an obvious projection

$$\text{pr}: H_1(G \ltimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \rightarrow H_1(G)_f.$$ 

Let

$$P(f) \in \mathcal{P}(\mathbb{R} \otimes \mathbb{Z} H_1(G)_f)$$

be the image of $\rho_u^{(2)}(f)$ under the composite

$$\text{Wh}^w(G \ltimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}(\mathbb{R} \otimes \mathbb{Z} H_1(G \ltimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}(\text{pr})} \mathcal{P}(\mathbb{R} \otimes \mathbb{Z} H_1(G)_f)$$

What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where $F$ is a finitely generated free group, is of particular interest.
Suppose that $H_1(f) = \text{id}$. Then there is an obvious projection

$$\text{pr}: H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \to H_1(G)_f.$$ 

Let

$$P(f) \in \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

be the image of $\rho^{(2)}_u(f)$ under the composite

$$\text{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}(\text{pr})} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f).$$

What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where $F$ is a finitely generated free group, is of particular interest.
The Thurston norm can also be read of from an $L^2$-Euler characteristic.

The higher order Alexander polynomials due to Harvey and Cochrane are special cases of the universal $L^2$-torsion and we can prove the conjecture that their degree is the Thurston norm.

The universal $L^2$-torsion seems to give an interesting invariant for elements in $\text{Out}(F_n)$ and mapping class groups.