Introduction to $L^2$-invariants

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

Fort Worth, June, 2015
Basic motivation

- Given an invariant for finite $CW$-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.

- Examples:

<table>
<thead>
<tr>
<th>Classical notion</th>
<th>generalized version</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homology with coefficients in $\mathbb{Z}$</td>
<td>Homology with coefficients in representations</td>
</tr>
<tr>
<td>Euler characteristic $\in \mathbb{Z}$</td>
<td>Walls finiteness obstruction in $K_0(\mathbb{Z}\pi)$</td>
</tr>
<tr>
<td>Lefschetz numbers $\in \mathbb{Z}$</td>
<td>Generalized Lefschetz invariants in $\mathbb{Z}\pi_\phi$</td>
</tr>
<tr>
<td>Signature $\in \mathbb{Z}$</td>
<td>Surgery invariants in $L_* (\mathbb{Z}G)$</td>
</tr>
<tr>
<td></td>
<td>torsion invariants</td>
</tr>
</tbody>
</table>
Basic motivation

- Given an invariant for finite \( CW \)-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.

Examples:

<table>
<thead>
<tr>
<th>Classical notion</th>
<th>generalized version</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homology with coefficients in ( \mathbb{Z} )</td>
<td>Homology with coefficients in representations</td>
</tr>
<tr>
<td>Euler characteristic ( \in \mathbb{Z} )</td>
<td>Walls finiteness obstruction in ( K_0(\mathbb{Z}\pi) )</td>
</tr>
<tr>
<td>Lefschetz numbers ( \in \mathbb{Z} )</td>
<td>Generalized Lefschetz invariants in ( \mathbb{Z}\pi\phi )</td>
</tr>
<tr>
<td>Signature ( \in \mathbb{Z} )</td>
<td>Surgery invariants in ( L_*(\mathbb{Z}G) )</td>
</tr>
</tbody>
</table>

— torsion invariants
Basic motivation

- Given an invariant for finite $CW$-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.

**Examples:**

<table>
<thead>
<tr>
<th>Classical notion</th>
<th>generalized version</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homology with coefficients in $\mathbb{Z}$</td>
<td>Homology with coefficients in representations</td>
</tr>
<tr>
<td>Euler characteristic $\in \mathbb{Z}$</td>
<td>Walls finiteness obstruction in $K_0(\mathbb{Z}\pi)$</td>
</tr>
<tr>
<td>Lefschetz numbers $\in \mathbb{Z}$</td>
<td>Generalized Lefschetz invariants in $\mathbb{Z}\pi_\phi$</td>
</tr>
<tr>
<td>Signature $\in \mathbb{Z}$</td>
<td>Surgery invariants in $L^*(\mathbb{Z}G)$</td>
</tr>
<tr>
<td>—</td>
<td>torsion invariants</td>
</tr>
</tbody>
</table>
We want to apply this principle to (classical) Betti numbers

\[ b_n(X) := \dim_{\mathbb{C}}(H_n(X; \mathbb{C})). \]

Here are two naive attempts which fail:

1. \( \dim_{\mathbb{C}}(H_n(\tilde{X}; \mathbb{C})) \)
2. \( \dim_{\mathbb{C}_\pi}(H_n(\tilde{X}; \mathbb{C})) \),

where \( \dim_{\mathbb{C}_\pi}(M) \) for a \( \mathbb{C}[\pi] \)-module could be chosen for instance as \( \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}G} M) \).

The problem is that \( \mathbb{C}_\pi \) is in general not Noetherian and \( \dim_{\mathbb{C}_\pi}(M) \) is in general not additive under exact sequences.

We will use the following successful approach which is essentially due to Atiyah.
We want to apply this principle to (classical) Betti numbers

\[ b_n(X) := \dim_{\mathbb{C}}(H_n(X; \mathbb{C})). \]

Here are two naive attempts which fail:

- \( \dim_{\mathbb{C}}(H_n(\tilde{X}; \mathbb{C})) \)
- \( \dim_{\mathbb{C}[\pi]}(H_n(\tilde{X}; \mathbb{C})) \),

where \( \dim_{\mathbb{C}[\pi]}(M) \) for a \( C[\pi] \)-module could be chosen for instance as \( \dim_{\mathbb{C}}(\mathbb{C} \otimes_{C[G]} M) \).

The problem is that \( C[\pi] \) is in general not Noetherian and \( \dim_{\mathbb{C}[\pi]}(M) \) is in general not additive under exact sequences.

We will use the following successful approach which is essentially due to Atiyah.
We want to apply this principle to (classical) Betti numbers

\[ b_n(X) := \dim_{\mathbb{C}}(H_n(X; \mathbb{C})). \]

Here are two naive attempts which fail:

- \( \dim_{\mathbb{C}}(H_n(\tilde{X}; \mathbb{C})) \)
- \( \dim_{\mathbb{C}[\pi]}(H_n(\tilde{X}; \mathbb{C})) \),

where \( \dim_{\mathbb{C}[\pi]}(M) \) for a \( \mathbb{C}[\pi] \)-module could be chosen for instance as \( \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C} G} M) \).

The problem is that \( \mathbb{C}[\pi] \) is in general not Noetherian and \( \dim_{\mathbb{C}[\pi]}(M) \) is in general not additive under exact sequences.

We will use the following successful approach which is essentially due to Atiyah.
Throughout these lectures let $G$ be a discrete group.

Given a ring $R$ and a group $G$, denote by $RG$ or $R[G]$ the group ring.

Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients $r_g$ are non-zero.

Addition is given by adding the coefficients.

Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$).

In general $RG$ is a very complicated ring.
Throughout these lectures let $G$ be a discrete group.

Given a ring $R$ and a group $G$, denote by $RG$ or $R[G]$ the group ring.

Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients $r_g$ are non-zero.

Addition is given by adding the coefficients.

Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$).

In general $RG$ is a very complicated ring.
Throughout these lectures let $G$ be a discrete group.

Given a ring $R$ and a group $G$, denote by $RG$ or $R[G]$ the group ring.

Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients $r_g$ are non-zero.

Addition is given by adding the coefficients.

Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$).

In general $RG$ is a very complicated ring.
Denote by \( L^2(G) \) the Hilbert space of (formal) sums \( \sum_{g \in G} \lambda_g \cdot g \) such that \( \lambda_g \in \mathbb{C} \) and \( \sum_{g \in G} |\lambda_g|^2 < \infty \).

**Definition**

Define the group von Neumann algebra

\[
\mathcal{N}(G) := \mathcal{B}(L^2(G)), \quad L^2(G)^G = \overline{\mathbb{C}G}^{\text{weak}}
\]

to be the algebra of bounded \( G \)-equivariant operators \( L^2(G) \rightarrow L^2(G) \).

The von Neumann trace is defined by

\[
\text{tr}_\mathcal{N}(G) : \mathcal{N}(G) \rightarrow \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.
\]

**Example (Finite \( G \))**

If \( G \) is finite, then \( \mathbb{C}G = L^2(G) = \mathcal{N}(G) \). The trace \( \text{tr}_\mathcal{N}(G) \) assigns to \( \sum_{g \in G} \lambda_g \cdot g \) the coefficient \( \lambda_e \).
Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

**Definition**

Define the group von Neumann algebra $\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G)^G = \mathbb{C}G)^{\text{weak}}$ to be the algebra of bounded $G$-equivariant operators $L^2(G) \to L^2(G)$.

The von Neumann trace is defined by

$$\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$ 

**Example (Finite $G$)**

If $G$ is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\text{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient $\lambda_e$. 

Wolfgang Lück (HIM, Bonn) Introduction to $L^2$-invariants Fort Worth, June, 2015 5 / 45
Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

**Definition**

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \mathbb{C}G^{\text{weak}}$$

to be the algebra of bounded $G$-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$ 

**Example (Finite $G$)**

If $G$ is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\text{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient $\lambda_e$. 
Example \((G = \mathbb{Z}^n)\)

Let \(G = \mathbb{Z}^n\). Let \(L^2(T^n)\) be the Hilbert space of \(L^2\)-integrable functions \(T^n \to \mathbb{C}\). Fourier transform yields an isometric \(\mathbb{Z}^n\)-equivariant isomorphism

\[ L^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n). \]

Let \(L^\infty(T^n)\) be the Banach space of essentially bounded measurable functions \(f : T^n \to \mathbb{C}\). We obtain an isomorphism

\[ L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f \]

where \(M_f : L^2(T^n) \to L^2(T^n)\) is the bounded \(\mathbb{Z}^n\)-operator \(g \mapsto g \cdot f\).

Under this identification the trace becomes

\[ \text{tr}_{\mathcal{N}(\mathbb{Z}^n)} : L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu. \]
von Neumann dimension

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists an isometric linear $G$-embedding of $V$ into $L^2(G)^n$ for some $n \geq 0$.

A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f: V \to W$ is a bounded $G$-equivariant operator.

Definition (von Neumann dimension)

Let $V$ be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a $G$-equivariant projection $p: L^2(G)^n \to L^2(G)^n$ with $\text{im}(p) \cong \mathcal{N}(G) V$.

Define the von Neumann dimension of $V$ by

$$\dim_{\mathcal{N}(G)}(V) := \text{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \text{tr}_{\mathcal{N}(G)}(p_{i,i}) \in [0, \infty).$$
von Neumann dimension

**Definition (Finitely generated Hilbert module)**

A finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists an isometric linear $G$-embedding of $V$ into $L^2(G)^n$ for some $n \geq 0$. A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f : V \to W$ is a bounded $G$-equivariant operator.

**Definition (von Neumann dimension)**

Let $V$ be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a $G$-equivariant projection $p : L^2(G)^n \to L^2(G)^n$ with $\text{im}(p) \cong_{\mathcal{N}(G)} V$. Define the von Neumann dimension of $V$ by

$$\dim_{\mathcal{N}(G)}(V) := \text{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \text{tr}_{\mathcal{N}(G)}(p_{i,i}) \in [0, \infty).$$
von Neumann dimension

**Definition (Finitely generated Hilbert module)**

A finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists an isometric linear $G$-embedding of $V$ into $L^2(G)^n$ for some $n \geq 0$. A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f: V \rightarrow W$ is a bounded $G$-equivariant operator.

**Definition (von Neumann dimension)**

Let $V$ be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a $G$-equivariant projection $p: L^2(G)^n \rightarrow L^2(G)^n$ with $\text{im}(p) \cong_{\mathcal{N}(G)} V$. Define the von Neumann dimension of $V$ by

$$\dim_{\mathcal{N}(G)}(V) := \text{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \text{tr}_{\mathcal{N}(G)}(p_{i,i}) \in [0, \infty).$$
Example (Finite $G$)

For finite $G$ a finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is the same as a unitary finite dimensional $G$-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example ($G = \mathbb{Z}^n$)

Let $G$ be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} : L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with $\chi_X$. Its image $V$ is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \text{vol}(X).$$

In particular each $r \in [0, \infty)$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$. 

Wolfgang Lück (HIM, Bonn) 
Introduction to $L^2$-invariants 

Fort Worth, June, 2015 

8 / 45
**Example (Finite $G$)**

For finite $G$ a finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is the same as a unitary finite dimensional $G$-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

---

**Example ($G = \mathbb{Z}^n$)**

Let $G$ be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} : L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with $\chi_X$. Its image $V$ is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \text{vol}(X).$$

In particular each $r \in [0, \infty)$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$. 
**Definition (Weakly exact)**

A sequence of Hilbert $\mathcal{N}(G)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$ is weakly exact at $V$ if the kernel $\ker(p)$ of $p$ and the closure $(\text{im}(i))$ of the image $\text{im}(i)$ of $i$ agree.

A map of Hilbert $\mathcal{N}(G)$-modules $f: V \rightarrow W$ is a weak isomorphism if it is injective and has dense image.

**Example**

The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

$$M_{z-1}: L^2(\mathbb{Z}) = L^2(S^1) \rightarrow L^2(\mathbb{Z}) = L^2(S^1), \quad u(z) \mapsto (z - 1) \cdot u$$

is a weak isomorphism, but not an isomorphism.
Definition (Weakly exact)

A sequence of Hilbert $\mathcal{N}(G)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$ is weakly exact at $V$ if the kernel $\ker(p)$ of $p$ and the closure $(\text{im}(i))$ of the image $\text{im}(i)$ of $i$ agree.

A map of Hilbert $\mathcal{N}(G)$-modules $f : V \to W$ is a weak isomorphism if it is injective and has dense image.

Example

The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

$$M_{z-1} : L^2(\mathbb{Z}) = L^2(S^1) \to L^2(\mathbb{Z}) = L^2(S^1), \quad u(z) \mapsto (z - 1) \cdot u$$

is a weak isomorphism, but not an isomorphism.
**Theorem (Main properties of the von Neumann dimension)**

1. **Faithfulness**

   We have for a finitely generated Hilbert $\mathcal{N}(G)$-module $V$

   
   \[ V = 0 \iff \dim_{\mathcal{N}(G)}(V) = 0; \]

2. **Additivity**

   If \( 0 \to U \to V \to W \to 0 \) is a weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules, then

   \[ \dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V); \]

3. **Cofinality**

   Let \( \{ V_i \mid i \in I \} \) be a directed system of Hilbert $\mathcal{N}(G)$-submodules of $V$, directed by inclusion. Then

   \[ \dim_{\mathcal{N}(G)} \left( \bigcup_{i \in I} V_i \right) = \sup \{ \dim_{\mathcal{N}(G)}(V_i) \mid i \in I \}. \]
Definition ($L^2$-homology and $L^2$-Betti numbers)

Let $X$ be a connected $CW$-complex of finite type. Let $\tilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_\ast(\tilde{X})$ its cellular $\mathbb{Z}_\pi$-chain complex.

Define its cellular $L^2$-chain complex to be the Hilbert $N(\pi)$-chain complex

$$C^{(2)}_\ast(\tilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}_\pi} C_\ast(\tilde{X}) = C_\ast(\tilde{X}).$$

Define its $n$-th $L^2$-homology to be the finitely generated Hilbert $N(G)$-module

$$H^{(2)}_n(\tilde{X}) := \ker(c^{(2)}_n)/\text{im}(c^{(2)}_{n+1}).$$

Define its $n$-th $L^2$-Betti number

$$b^{(2)}_n(\tilde{X}) := \dim_{N(\pi)}(H^{(2)}_n(\tilde{X})) \in \mathbb{R}_{\geq 0}. $$
Definition ($L^2$-homology and $L^2$-Betti numbers)

Let $X$ be a connected CW-complex of finite type. Let $\tilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_\ast(\tilde{X})$ its cellular $\mathbb{Z}_\pi$-chain complex. Define its cellular $L^2$-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C^{(2)}_\ast(\tilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}_\pi} C_\ast(\tilde{X}) = C_\ast(\tilde{X}).$$

Define its $n$-th $L^2$-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H^{(2)}_n(\tilde{X}) := \ker(c^{(2)}_n)/\text{im}(c^{(2)}_{n+1}).$$

Define its $n$-th $L^2$-Betti number

$$b^{(2)}_n(\tilde{X}) := \dim_{\mathcal{N}(\pi)} (H^{(2)}_n(\tilde{X})) \in \mathbb{R}_{\geq 0}.$$
Theorem (Main properties of Betti numbers)

Let $X$ and $Y$ be connected CW-complexes of finite type.

- **Homotopy invariance**
  
  If $X$ and $Y$ are homotopy equivalent, then
  
  $$b_n(X) = b_n(Y);$$

- **Euler-Poincaré formula**
  
  We have
  
  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n(X);$$

- **Poincaré duality**
  
  Let $M$ be a closed manifold of dimension $d$. Then
  
  $$b_n(M) = b_{d-n}(M);$$
Theorem (Main properties of $L^2$-Betti numbers)

Let $X$ and $Y$ be connected CW-complexes of finite type.

- **Homotopy invariance**
  If $X$ and $Y$ are homotopy equivalent, then
  \[
  b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\tilde{Y});
  \]

- **Euler-Poincaré formula**
  We have
  \[
  \chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\tilde{X});
  \]

- **Poincaré duality**
  Let $M$ be a closed manifold of dimension $d$. Then
  \[
  b_n^{(2)}(\tilde{M}) = b_{d-n}^{(2)}(\tilde{M});
  \]
Theorem (Continued)

- **Künneth formula**

\[
b_n(X \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y);
\]

- **Zero-th \(L^2\)-Betti number**

We have

\[
b_0(X) = 1;
\]
Theorem (Continued)

- **Künneth formula**

\[
b_n^{(2)}(\tilde{X} \times \tilde{Y}) = \sum_{p+q=n} b_p^{(2)}(\tilde{X}) \cdot b_q^{(2)}(\tilde{Y});
\]

- **Zero-th \(L^2\)-Betti number**

  We have

\[
b_0^{(2)}(\tilde{X}) = \frac{1}{|\pi|};
\]
Theorem (Continued)

- **Künneth formula**

\[
b^{(2)}_n(\widetilde{X} \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y);
\]

- **Zero-th \(L^2\)-Betti number**

We have

\[
b^{(2)}_0(\widetilde{X}) = \frac{1}{|\pi|};
\]

- **Finite coverings**

If \(X \to Y\) is a finite covering with \(d\) sheets, then

\[
b^{(2)}_n(\widetilde{X}) = d \cdot b^{(2)}_n(\widetilde{Y}).
\]
Example (Finite $\pi$)

If $\pi$ is finite then

$$b_n^{(2)}(\widetilde{X}) = \frac{b_n(\widetilde{X})}{|\pi|}.$$ 

Example ($S^1$)

Consider the $\mathbb{Z}$-CW-complex $\widetilde{S^1}$. We get for $C_*^{(2)}(\widetilde{S^1})$

$$\ldots \rightarrow 0 \rightarrow L^2(\mathbb{Z}) \xrightarrow{M_{z^{-1}}} L^2(\mathbb{Z}) \rightarrow 0 \rightarrow \ldots$$

and hence $H_n^{(2)}(\widetilde{S^1}) = 0$ and $b_n^{(2)}(\widetilde{S^1}) = 0$ for all $n \geq 0$. 
Example (Finite $\pi$)

If $\pi$ is finite then

$$b_n^{(2)}(\tilde{X}) = \frac{b_n(\tilde{X})}{|\pi|}.$$ 

Example ($S^1$)

Consider the $\mathbb{Z}$-CW-complex $\tilde{S}^1$. We get for $C_*^{(2)}(\tilde{S}^1)$

$$\cdots \to 0 \to L^2(\mathbb{Z}) \xrightarrow{M_{z^{-1}}} L^2(\mathbb{Z}) \to 0 \to \cdots$$

and hence $H_n^{(2)}(\tilde{S}^1) = 0$ and $b_n^{(2)}(\tilde{S}^1) = 0$ for all $n \geq 0$. 

Wolfgang Lück (HIM, Bonn)
Example \((\pi = \mathbb{Z}^d)\)

Let \(X\) be a connected \(CW\)-complex of finite type with fundamental group \(\mathbb{Z}^d\). Let \(\mathbb{C}[\mathbb{Z}^d]^{(0)}\) be the quotient field of the commutative integral domain \(\mathbb{C}[\mathbb{Z}^d]\). Then

\[
b_n^{(2)}(\tilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left( \mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\tilde{X}) \right)
\]

Obviously this implies

\[
b_n^{(2)}(\tilde{X}) \in \mathbb{Z}.
\]
Some computations and results

Example (Finite self coverings)

We get for a connected CW-complex \( X \) of finite type, for which there is a selfcovering \( X \to X \) with \( d \)-sheets for some integer \( d \geq 2 \),

\[
\beta_n^{(2)}(\tilde{X}) = 0 \quad \text{for } n \geq 0.
\]

This implies for each connected CW-complex \( Y \) of finite type

\[
\beta_n^{(2)}(S^1 \times Y) = 0 \quad \text{for } n \geq 0.
\]
Theorem ($S^1$-actions, Lück)

Let $M$ be a connected compact manifold with $S^1$-action. Suppose that for one (and hence all) $x \in X$ the map $S^1 \to X$, $z \mapsto zx$ is $\pi_1$-injective. Then we get for all $n \geq 0$

$$b_n^{(2)}(\tilde{X}) = 0.$$ 

Theorem ($S^1$-actions on aspherical manifolds, Lück)

Let $M$ be an aspherical closed manifold with non-trivial $S^1$-action. Then

1. The action has no fixed points;
2. The map $S^1 \to X$, $z \mapsto zx$ is $\pi_1$-injective for $x \in X$;
3. $b_n^{(2)}(\tilde{M}) = 0$ for $n \geq 0$ and $\chi(M) = 0$. 

Wolfgang Lück (HIM, Bonn) 
Introduction to $L^2$-invariants 
Fort Worth, June, 2015 17 / 45
Theorem ($S^1$-actions, Lück)

Let $M$ be a connected compact manifold with $S^1$-action. Suppose that for one (and hence all) $x \in X$ the map $S^1 \to X$, $z \mapsto zx$ is $\pi_1$-injective. Then we get for all $n \geq 0$

$$b_n^{(2)}(\widetilde{X}) = 0.$$ 

Theorem ($S^1$-actions on aspherical manifolds, Lück)

Let $M$ be an aspherical closed manifold with non-trivial $S^1$-action. Then

1. The action has no fixed points;
2. The map $S^1 \to X$, $z \mapsto zx$ is $\pi_1$-injective for $x \in X$;
3. $b_n^{(2)}(\widetilde{M}) = 0$ for $n \geq 0$ and $\chi(M) = 0$. 

Wolfgang Lück (HIM, Bonn) 

Introduction to $L^2$-invariants 

Fort Worth, June, 2015 17 / 45
Example ($L^2$-Betti number of surfaces)

- Let $F_g$ be the orientable closed surface of genus $g \geq 1$.
- Then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$.
- By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$.
- $\dim(F_g) = 2$, we get $b_n^{(2)}(\widetilde{F_g}) = 0$ for $n \geq 3$.
- The Euler-Poincaré formula shows

\[
    b_1^{(2)}(\widetilde{F_g}) = -\chi(F_g) = 2g - 2;
\]
\[
    b_n^{(2)}(\widetilde{F_0}) = 0 \quad \text{for } n \neq 1.
\]
Theorem (Hodge - de Rham Theorem)

Let $M$ be a closed Riemannian manifold. Put

$$\mathcal{H}^n(M) = \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = 0\}$$

Then integration defines an isomorphism of real vector spaces

$$\mathcal{H}^n(M) \cong H^n(M; \mathbb{R}).$$

Corollary (Betti numbers and heat kernels)

$$b_n(M) = \lim_{t \to \infty} \int_M \text{tr}_{\mathbb{R}}(e^{-t\Delta_n}(x, x)) \, d\text{vol}.$$  
where $e^{-t\Delta_n}(x, y)$ is the heat kernel on $M$. 

Wolfgang Lück (HIM, Bonn)  
Introduction to $L^2$-invariants  
Fort Worth, June, 2015  19 / 45
**Theorem (\(L^2\)-Hodge - de Rham Theorem, Dodziuk)**

Let \(M\) be a closed Riemannian manifold. Put

\[ H^n_{(2)}(\tilde{M}) = \{ \tilde{\omega} \in \Omega^n(\tilde{M}) \mid \tilde{\Delta}_n(\tilde{\omega}) = 0, \ ||\tilde{\omega}||_{L^2} < \infty \} \]

Then integration defines an isomorphism of finitely generated Hilbert \(N(\pi)\)-modules

\[ H^n_{(2)}(\tilde{M}) \cong H^n_{(2)}(\tilde{M}). \]

**Corollary (\(L^2\)-Betti numbers and heat kernels)**

\[ b^{(2)}_n(\tilde{M}) = \lim_{t \to \infty} \int_{F} \text{tr}_{\mathbb{R}}(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{y})) \ d\text{vol}. \]

where \(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{y})\) is the heat kernel on \(\tilde{M}\) and \(F\) is a fundamental domain for the \(\pi\)-action.
Theorem (hyperbolic manifolds, Dodziuk)

Let $M$ be a hyperbolic closed Riemannian manifold of dimension $d$. Then:

$$b_n^{(2)}(\tilde{M}) = \begin{cases} 
= 0 & \text{if } 2n \neq d; \\
> 0 & \text{if } 2n = d.
\end{cases}$$

Proof.

A direct computation shows that $\mathcal{H}_{(2)}^p(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that $M$ is hyperbolic if and only if $\tilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. 

Wolfgang Lück (HIM, Bonn) 

Introduction to $L^2$-invariants 

Fort Worth, June, 2015
Theorem (hyperbolic manifolds, Dodziuk)

Let $M$ be a hyperbolic closed Riemannian manifold of dimension $d$. Then:

$$b_n^{(2)}(\tilde{M}) = \begin{cases} 
0, & \text{if } 2n \neq d; \\
> 0, & \text{if } 2n = d.
\end{cases}$$

Proof.

A direct computation shows that $\mathcal{H}_{(2)}^p(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that $M$ is hyperbolic if and only if $\tilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. \qed
Corollary

Let $M$ be a hyperbolic closed manifold of dimension $d$. Then

1. If $d = 2m$ is even, then

\[ (-1)^m \cdot \chi(M) > 0; \]

2. $M$ carries no non-trivial $S^1$-action.

Proof.

(1) We get from the Euler-Poincaré formula and the last result

\[ (-1)^m \cdot \chi(M) = b_m^{(2)}(\tilde{M}) > 0. \]

(2) We give the proof only for $d = 2m$ even. Then $b_m^{(2)}(\tilde{M}) > 0$. Since $\tilde{M} = \mathbb{H}^d$ is contractible, $M$ is aspherical. Now apply a previous result about $S^1$-actions.
Corollary

Let $M$ be a hyperbolic closed manifold of dimension $d$. Then

1. If $d = 2m$ is even, then

   $$(-1)^m \cdot \chi(M) > 0;$$

2. $M$ carries no non-trivial $S^1$-action.

Proof.

(1) We get from the Euler-Poincaré formula and the last result

   $$(-1)^m \cdot \chi(M) = b_m^{(2)}(\tilde{M}) > 0.$$

(2) We give the proof only for $d = 2m$ even. Then $b_m^{(2)}(\tilde{M}) > 0$. Since $\tilde{M} = \mathbb{H}^d$ is contractible, $M$ is aspherical. Now apply a previous result about $S^1$-actions.
Theorem (3-manifolds, Lott-Lück)

Let the 3-manifold $M$ be the connected sum $M_1 \# \ldots \# M_r$ of (compact connected orientable) prime 3-manifolds $M_j$. Assume that $\pi_1(M)$ is infinite. Then

$$b_1^{(2)}(\tilde{M}) = (r - 1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} - \chi(M)$$

$$+ \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right| ;$$

$$b_2^{(2)}(\tilde{M}) = (r - 1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|}$$

$$+ \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right| ;$$

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{for } n \neq 1, 2.$$
Theorem (mapping tori, Lück)

Let $f : X \to X$ be a cellular selfhomotopy equivalence of a connected $CW$-complex $X$ of finite type. Let $T_f$ be the mapping torus. Then

$$b_n^{(2)}(\tilde{T}_f) = 0 \quad \text{for } n \geq 0.$$
Proof:

- As $T_{fd} \to T_f$ is a $d$-sheeted covering, we get
  $$b_n^{(2)}(\widetilde{T}_f) = \frac{b_n^{(2)}(\widetilde{T}_{fd})}{d}.$$ 

- If $\beta_n(X)$ is the number of $n$-cells, then there is up to homotopy equivalence a $CW$-structure on $T_{fd}$ with $\beta_n(T_{fd}) = \beta_n(X) + \beta_{n-1}(X)$. We have
  $$b_n^{(2)}(\widetilde{T}_{fd}) = \dim_{\mathcal{N}(G)} \left( H_n^{(2)}(C_n^{(2)}(\widetilde{T}_{fd})) \right) \leq \dim_{\mathcal{N}(G)} \left( C_n^{(2)}(\widetilde{T}_{fd}) \right) = \beta_n(T_{fd}).$$

- This implies for all $d \geq 1$
  $$b_n^{(2)}(\widetilde{T}_f) \leq \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$ 

- Taking the limit for $d \to \infty$ yields the claim.
Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let $G$ be a torsionfree finitely presented group. We say that $G$ satisfies the Atiyah Conjecture if for any closed Riemannian manifold $M$ with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\tilde{M}) \in \mathbb{Z}.$$ 

All computations presented above support the Atiyah Conjecture.
Let $G$ be a torsionfree finitely presented group. We say that $G$ satisfies the Atiyah Conjecture if for any closed Riemannian manifold $M$ with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\tilde{M}) \in \mathbb{Z}.$$
Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let $G$ be a torsionfree finitely presented group. We say that $G$ satisfies the Atiyah Conjecture if for any closed Riemannian manifold $M$ with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\tilde{M}) \in \mathbb{Z}.$$

All computations presented above support the Atiyah Conjecture.
The **fundamental square** is given by the following inclusions of rings

$$\begin{array}{ccc}
\mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\
\downarrow & & \downarrow \\
\mathcal{D}(G) & \longrightarrow & \mathcal{U}(G)
\end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the **Ore localization** of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.
The fundamental square is given by the following inclusions of rings

\[
\begin{array}{ccc}
\mathbb{Z}G & \rightarrow & \mathcal{N}(G) \\
\downarrow & & \downarrow \\
\mathcal{D}(G) & \rightarrow & \mathcal{U}(G)
\end{array}
\]

- \(\mathcal{U}(G)\) is the algebra of affiliated operators. Algebraically it is just the Ore localization of \(\mathcal{N}(G)\) with respect to the multiplicatively closed subset of non-zero divisors.
- \(\mathcal{D}(G)\) is the division closure of \(\mathbb{Z}G\) in \(\mathcal{U}(G)\), i.e., the smallest subring of \(\mathcal{U}(G)\) containing \(\mathbb{Z}G\) such that every element in \(\mathcal{D}(G)\), which is a unit in \(\mathcal{U}(G)\), is already a unit in \(\mathcal{D}(G)\) itself.
If $G$ is finite, it is given by

$$\mathbb{Z}G \longrightarrow \mathbb{C}G \quad \downarrow \quad \text{id}$$

$$\mathbb{Q}G \longrightarrow \mathbb{C}G$$

If $G = \mathbb{Z}$, it is given by

$$\mathbb{Z}[\mathbb{Z}] \longrightarrow L^\infty(S^1)$$

$$\mathbb{Q}[\mathbb{Z}]^{(0)} \longrightarrow L(S^1)$$
• If $G$ is finite, its is given by

$$\begin{array}{c}
\mathbb{Z}G \\ \downarrow \\
\mathbb{Q}G
\end{array} \rightarrow 
\begin{array}{c}
\mathbb{C}G \\ \downarrow \text{id}
\end{array}
\begin{array}{c}
\mathbb{Q}G \\ \downarrow
\end{array} \rightarrow 
\begin{array}{c}
\mathbb{C}G
\end{array}$$

• If $G = \mathbb{Z}$, it is given by

$$\begin{array}{c}
\mathbb{Z}[\mathbb{Z}] \\ \downarrow \\
\mathbb{Q}[\mathbb{Z}]^{(0)}
\end{array} \rightarrow 
\begin{array}{c}
L^\infty(S^1) \\ \downarrow
\end{array}
\begin{array}{c}
\mathbb{Q}[\mathbb{Z}]^{(0)} \\ \downarrow
\end{array} \rightarrow 
\begin{array}{c}
L(S^1)
\end{array}$$
If $G$ is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.
If $G$ is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.
Conjecture (Atiyah Conjecture for torsionfree groups)

Let $G$ be a torsionfree group. It satisfies the Atiyah Conjecture if $D(G)$ is a skew-field.

- A torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension
  \[ \dim_{\mathcal{N}(G)} \left( \ker \left( r_A : \mathcal{N}(G)^m \to \mathcal{N}(G)^n \right) \right) \]
  is an integer. In this case this dimension agrees with
  \[ \dim_{\mathcal{D}(G)} \left( r_A : \mathcal{D}(G)^m \to \mathcal{D}(G)^n \right). \]

- The general version above is equivalent to the one stated before if $G$ is finitely presented.
Conjecture (Atiyah Conjecture for torsionfree groups)

Let $G$ be a torsionfree group. It satisfies the Atiyah Conjecture if $D(G)$ is a skew-field.

- A torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension
  \[
  \dim_{\mathcal{N}(G)} \left( \ker \left( r_A : \mathcal{N}(G)^m \to \mathcal{N}(G)^n \right) \right)
  \]
  is an integer. In this case this dimension agrees with
  \[
  \dim_{D(G)} \left( r_A : D(G)^m \to D(G)^n \right).
  \]

- The general version above is equivalent to the one stated before if $G$ is finitely presented.
Conjecture (Atiyah Conjecture for torsionfree groups)

Let $G$ be a torsionfree group. It satisfies the Atiyah Conjecture if $D(G)$ is a skew-field.

- A torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

$$\dim_{\mathcal{N}(G)}(\ker(r_A : \mathcal{N}(G)^m \to \mathcal{N}(G)^n))$$

is an integer. In this case this dimension agrees with

$$\dim_{\mathcal{D}(G)}(r_A : \mathcal{D}(G)^m \to \mathcal{D}(G)^n).$$

- The general version above is equivalent to the one stated before if $G$ is finitely presented.
The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero $F$ the group ring $FG$ has no non-trivial zero-divisors.

There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

However, there exist closed Riemannian manifolds whose universal coverings have an $L^2$-Betti number which is irrational, see Austin, Grabowski.
The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero $F$ the group ring $FG$ has no non-trivial zero-divisors.

There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

However, there exist closed Riemannian manifolds whose universal coverings have an $L^2$-Betti number which is irrational, see Austin, Grabowski.
The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero $F$ the group ring $FG$ has no non-trivial zero-divisors.

There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

However, there exist closed Riemannian manifolds whose universal coverings have an $L^2$-Betti number which is irrational, see Austin, Grabowski.
Theorem (Linnell, Schick)

1. Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group $G$ which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.

2. If $G$ is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.
Theorem (Linnell, Schick)

1. Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group $G$ which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.

2. If $G$ is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.
Strategy to prove the Atiyah Conjecture

1. Show that $K_0(\mathbb{C}) \rightarrow K_0(\mathbb{C}G)$ is surjective
   (This is implied by the Farrell-Jones Conjecture)

2. Show that $K_0(\mathbb{C}G) \rightarrow K_0(\mathcal{D}(G))$ is surjective.

3. Show that $\mathcal{D}(G)$ is semisimple.
Approximation

In general there are no relations between the Betti numbers $b_n(X)$ and the $L^2$-Betti numbers $b_n^{(2)}(\tilde{X})$ for a connected $CW$-complex $X$ of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\tilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$
Given an integer \( l \geq 1 \) and a sequence \( r_1, r_2, \ldots, r_l \) of non-negative rational numbers, we can construct a group \( G \) such that \( BG \) is of finite type and

\[
\begin{align*}
b_n^{(2)}(BG) &= r_n \quad \text{for } 1 \leq n \leq l; \\
b_n^{(2)}(BG) &= 0 \quad \text{for } l + 1 \leq n; \\
b_n(BG) &= 0 \quad \text{for } n \geq 1.
\end{align*}
\]

For any sequence \( s_1, s_2, \ldots \) of non-negative integers there is a \( CW \)-complex \( X \) of finite type such that for \( n \geq 1 \)

\[
\begin{align*}
b_n(X) &= s_n; \\
b_n^{(2)}(\tilde{X}) &= 0.
\end{align*}
\]
Theorem (Approximation Theorem, Lück)

Let $X$ be a connected CW-complex of finite type. Suppose that $\pi$ is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\bigcap_{i \geq 1} G_i = \{1\}$. Let $X_i$ be the finite $[\pi : G_i]$-sheeted covering of $X$ associated to $G_i$.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\tilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$
Ordinary Betti numbers are not multiplicative under finite coverings, whereas the $L^2$-Betti numbers are. With the expression

$$\lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}$$

we try to force the Betti numbers to be multiplicative by a limit process.

The theorem above says that $L^2$-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.
Applications to deficiency and signature

Definition (Deficiency)

Let $G$ be a finitely presented group. Define its deficiency

$$\text{defi}(G) := \max\{g(P) - r(P)\}$$

where $P$ runs over all presentations $P$ of $G$ and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation $P$. 
Let $G$ be a finitely presented group. Define its **deficiency**

\[
\text{defi}(G) := \max\{g(P) - r(P)\}
\]

where $P$ runs over all presentations $P$ of $G$ and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation $P$. 

**Definition (Deficiency)**
The free group $F_g$ has the obvious presentation $\langle s_1, s_2, \ldots, s_g \mid \emptyset \rangle$ and its deficiency is realized by this presentation, namely $\text{defi}(F_g) = g$.

If $G$ is a finite group, $\text{defi}(G) \leq 0$.

The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n \rangle$ realizes the deficiency.

The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is $-1$, the obvious presentation $\langle s, t \mid s^n, t^n, [s, t] \rangle$ realizes the deficiency.
Example (deficiency and free products)

The deficiency is not additive under free products by the following example due to Hog-Lustig-Metzler. The group

\[(\mathbb{Z}/2 \times \mathbb{Z}/2) \ast (\mathbb{Z}/3 \times \mathbb{Z}/3)\]

has the obvious presentation

\[\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle\]

One may think that its deficiency is $-2$. However, it turns out that its deficiency is $-1$ realized by the following presentation

\[\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.\]
Lemma

Let $G$ be a finitely presented group. Then

$$\text{defi}(G) \leq 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation $P$ that

$$g(P) - r(P) \leq 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let $X$ be a $CW$-complex realizing $P$. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\tilde{X}) + b_1^{(2)}(\tilde{X}) - b_2^{(2)}(\tilde{X}).$$

Since the classifying map $X \rightarrow BG$ is 2-connected, we get

$$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(G) \quad \text{for } n = 0, 1;$$

$$b_2^{(2)}(\tilde{X}) \geq b_2^{(2)}(G).$$
Lemma

Let $G$ be a finitely presented group. Then

$$\text{defi}(G) \leq 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation $P$ that

$$g(P) - r(P) \leq 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let $X$ be a $CW$-complex realizing $P$. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\tilde{X}) + b_1^{(2)}(\tilde{X}) - b_2^{(2)}(\tilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(G) \quad \text{for} \quad n = 0, 1;$$

$$b_2^{(2)}(\tilde{X}) \geq b_2^{(2)}(G).$$
Theorem (Deficiency and extensions, Lück)

Let \( 1 \to H \overset{i}{\to} G \overset{q}{\to} K \to 1 \) be an exact sequence of infinite groups. Suppose that \( G \) is finitely presented \( H \) is finitely generated. Then:

1. \( b_1^{(2)}(G) = 0 \);
2. \( \text{defi}(G) \leq 1 \);
3. Let \( M \) be a closed oriented 4-manifold with \( G \) as fundamental group. Then
   \[
   | \text{sign}(M) | \leq \chi(M).
   \]
The Singer Conjecture

Conjecture (Singer Conjecture)

If $M$ is an aspherical closed manifold, then

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If $M$ is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\tilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.
- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.
The Singer Conjecture

If $M$ is an aspherical closed manifold, then

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If $M$ is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\tilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

The computations presented above do support the Singer Conjecture.

Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.
The Singer Conjecture

Conjecture (Singer Conjecture)

If \( M \) is an aspherical closed manifold, then

\[
b_n^{(2)}(\tilde{M}) = 0 \quad \text{if} \ 2n \neq \dim(M).
\]

If \( M \) is a closed Riemannian manifold with negative sectional curvature, then

\[
b_n^{(2)}(\tilde{M}) \begin{cases} 
= 0 & \text{if} \ 2n \neq \dim(M); \\
> 0 & \text{if} \ 2n = \dim(M).
\end{cases}
\]

● The computations presented above do support the Singer Conjecture.

● Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.
Because of the Euler-Poincaré formula

\[ \chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^2(\tilde{M}) \]

the Singer Conjecture implies the following conjecture provided that \( M \) has non-positive sectional curvature.

**Conjecture (Hopf Conjecture)**

*If \( M \) is a closed Riemannian manifold of even dimension with sectional curvature \( \sec(M) \), then*

\[
\begin{align*}
(-1)^{\dim(M)/2} \cdot \chi(M) &> 0 \quad \text{if} \quad \sec(M) < 0; \\
(-1)^{\dim(M)/2} \cdot \chi(M) &\geq 0 \quad \text{if} \quad \sec(M) \leq 0; \\
\chi(M) &= 0 \quad \text{if} \quad \sec(M) = 0; \\
\chi(M) &\geq 0 \quad \text{if} \quad \sec(M) \geq 0; \\
\chi(M) &> 0 \quad \text{if} \quad \sec(M) > 0.
\end{align*}
\]
Definition (Kähler hyperbolic manifold)

A Kähler hyperbolic manifold is a closed connected Kähler manifold $M$ whose fundamental form $\omega$ is $\tilde{d}$-bounded, i.e. its lift $\tilde{\omega} \in \Omega^2(\tilde{M})$ to the universal covering can be written as $d(\eta)$ holds for some bounded 1-form $\eta \in \Omega^1(\tilde{M})$.

Theorem (Gromov)

Let $M$ be a closed Kähler hyperbolic manifold of complex dimension $c$. Then

\[ b_n^{(2)}(\tilde{M}) = 0 \quad \text{if} \ n \neq c; \]
\[ b_n^{(2)}(\tilde{M}) > 0; \]
\[ (-1)^m \cdot \chi(M) > 0; \]
Let $M$ be a closed Kähler manifold. It is Kähler hyperbolic if it admits some Riemannian metric with negative sectional curvature, or, if, generally $\pi_1(M)$ is word-hyperbolic and $\pi_2(M)$ is trivial.

A consequence of the theorem above is that any Kähler hyperbolic manifold is a projective algebraic variety.