Universal torsion, L^2-invariants, polytopes and the Thurston norm

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Fort Worth, June, 2015

Review of classical L^2-invariants

- Given a group G, its group von Neumann algebra is defined to be

$$\mathcal{N}(G) = \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}.$$

- It comes with a natural trace

$$\operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e\rangle_{L^2(G)}.$$

- For $x = \sum_{g\in G} \lambda_g \cdot g \in \mathbb{C}G$ we have $\operatorname{tr}_{\mathcal{N}(G)}(x) = \lambda_e$.
- If G is finite, $\mathcal{N}(G) = \mathbb{C}G$.
- If $G = \mathbb{Z}$, we get identifications

$$\begin{aligned} \mathcal{N}(\mathbb{Z}) &= L^\infty(S^1); \\ \operatorname{tr}_{\mathcal{N}(\mathbb{Z})}(f) &= \int_{S^1} f(z)d\mu. \end{aligned}$$

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- The trace yields via the Hattori-Stallings rank the dimension function with values in $[0,\infty)$ for finitely generated projective $\mathcal{N}(G)$-modules due to Murray-von Neumann.

- This has been extended to the dimension function $\dim_{\mathcal{N}(G)}$ with values in $[0,\infty]$ for arbitrary $\mathcal{N}(G)$-modules, which has still nice properties such as additivity and cofinality, by Lück.

Definition (L^2-Betti number)

Let Y be a G-space. Define its nth L^2-Betti number

$$b_n^{(2)}(Y;\mathcal{N}(G)) := \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G)\otimes_{\mathbb{Z}G} C_*^{\mathrm{sing}}(Y))\big) \in [0,\infty].$$

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- This definition is the end of a long chain of generalizations of the original notion due to *Atiyah* which was motivated by index theory. He defined for a G-covering $\overline{M} \to M$ of a closed Riemannian manifold

$$b_n^{(2)}(\overline{M}) := \lim_{t \to \infty} \int_{\mathcal{F}} tr\left(e^{-t \cdot \overline{\Delta}_n}(\overline{x}, \overline{x})\right) dvol_{\overline{M}}.$$

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$$b_n^{(2)}(\overline{X}) = \frac{1}{|G|} \cdot b_n(\overline{X}).$$

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- In the sequel 3-manifold means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

Theorem (Lott-Lück)

For every 3-manifold M all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ vanish.

- We are interested in case where all L^2-Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called L^2-torsion.

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- L^2-torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing the notion of analytic Ray-Singer torsion. It can also be defined in terms of the cellular $\mathbb{Z}G$-chain complex, generalizing of the Reidemeister torsion.

- The definition of L^2-torsion is based on the notion of the Fuglede-Kadison determinant which is a generalization of the classical determinant to the infinite-dimensional setting. It is defined for an element $f \in \mathcal{N}(G)$ to be the non-negative real number

$$\det^{(2)}(f) = \frac{1}{2} \cdot \int \ln(\lambda)\, d\nu_{f^*f} \in \mathbb{R}$$

where ν_{f^*f} is the spectral measure of the positive operator f^*f.

- If G is finite, then $\det^{(2)}(f) = |G|^{-1} \cdot \ln(|\det(f)|)$.

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Theorem (Lück-Schick)

Let M be a 3-manifold. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

Definition ($K_1^w(\mathbb{Z}G)$)

Let $K_1^w(\mathbb{Z}G)$ be the abelian group given by:

- generators

  If $f\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$ is a $\mathbb{Z}G$-map such that the induced $\mathcal{N}(G)$-map $\mathcal{N}(G)^m \to \mathcal{N}(G)^m$ map is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator $[f]$ in $K_1^w(\mathbb{Z}G)$.
- relations

$$\begin{aligned} \left[\begin{pmatrix} f_1 & * \\ 0 & f_2 \end{pmatrix}\right] &= [f_1] + [f_2]; \\ [g \circ f] &= [f] + [g]. \end{aligned}$$

Define $\mathrm{Wh}^w(G) := K_1^w(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

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Let $G \to \overline{X} \to X$ be a G-covering of a finite CW-complex. Suppose that $\overline{X}$ is L^2-acyclic, i.e., $b_n^{(2)}(\overline{X})$ vanishes for all $n \in \mathbb{Z}$.

Then its universal L^2-torsion is defined as an element

$$\rho_u^{(2)}(\overline{X}) = \rho_u^{(2)}(\overline{X}; \mathcal{N}(G)) \in K_1^w(\mathbb{Z}G).$$

- We do not present the actual definition but it is similar to the definition of Whitehead torsion in terms of chain contractions, one has now to use weak chain contractions.
- It has indeed a universal property on the level of weakly acyclic finite based free $\mathbb{Z}G$-chain complexes.
- By stating its main properties and its relation to classical invariant we want to convince the audience that this is a very interesting and powerful invariant which is worthwhile to be explored further.

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- Homotopy invariance

Consider a pullback of G-coverings of finite CW-complexes with f a homotopy equivalence

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose that $\overline{Y}$ is L^2-acyclic.

Then $\overline{X}$ is L^2-acyclic and the obvious homomorphism

$$\mathrm{Wh}(G) \to \mathrm{Wh}^w(G)$$

sends the Whitehead torsion $\tau(\overline{f} \colon \overline{X} \to \overline{Y})$ to the difference of universal L^2-torsions $\rho_u^{(2)}(\overline{Y}) - \rho_u^{(2)}(\overline{X})$.

- Sum formula

  Let $G \to \overline{X} \to X$ be a G-covering of a finite CW-complex X. Let $X_i \subseteq X$ for $i = 0, 1, 2$ CW-subcomplexes with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Let $\overline{X_i} \to X_i$ be given by restriction. Suppose that $\overline{X_i}$ is L^2-acyclic for $i = 0, 1, 2$.

  Then $\overline{X}$ is L^2-acyclic and we get

$$\rho_u^{(2)}(\overline{X}; \mathcal{N}(G)) = \rho_u^{(2)}(\overline{X_1}; \mathcal{N}(G)) + \rho_u^{(2)}(\overline{X_2}; \mathcal{N}(G)) - \rho_u^{(2)}(\overline{X_0}; \mathcal{N}(G)).$$

- There are product formulas or more general formulas for fibrations and for proper S^1-actions which we will not state.
- Whenever an invariant comes from the universal torsion by applying a group homomorphism, these formulas automatically extend to the other invariant.

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Example (Trivial G)

- Suppose that G is trivial. Then we get isomorphisms

$$K_1^w(\{1\}) \xrightarrow{\cong} K_1(\mathbb{Q}) \xrightarrow{\det} \mathbb{Q}^\times$$
$$\mathrm{Wh}^w(\{1\}) \xrightarrow{\cong} \{r \in \mathbb{Q} \mid r > 0\}.$$

- A finite based free $\mathbb{Z}$-chain complex C_* is L^2-acyclic if and only if each of its homology groups is finite. Under the isomorphism above the universal torsion of C_* is sent to $\prod_{n \geq 0} |H_n(C_*)|^{(-1)^n}$.
- For finite CW-complexes we essentially have the same information as the Milnor torsion.

- If G is finite, we rediscover essentially the classical Reidemeister torsion.

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The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

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- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

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Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group G satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

$$\dim_{\mathcal{N}(G)} \left(\ker\left(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n\right)\right)$$

is an integer. In this case this dimension agrees with

$$\dim_{\mathcal{D}(G)}\left(r_A \colon \mathcal{D}(G)^m \to \mathcal{D}(G)^n\right).$$

- The Atiyah Conjecture implies the Kaplansky Conjecture saying that for any torsionfree group and field of characteristic zero F the group ring FG has no non-trivial zero-divisors.

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- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M};) \in \mathbb{Z}$ for every $n \geq 0$.

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Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.*
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

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Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück)

If G belongs to $\mathcal{C}$, then the natural map

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is an isomorphism.

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- In particular we get for $G = \mathbb{Z}$

$$K_1^w(\mathcal{D}(\mathbb{Z})) \cong \mathbb{Q}[\mathbb{Z}]^{(0)} \setminus \{0\}.$$

- It turns out that then the universal torsion is the same as the Alexander polynomial of an infinite cyclic covering, as it occurs for instance in knot theory.

Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by
$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$
- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion
$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$
- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

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- Thus we obtain the ϕ-twisted L^2-torsion function

$$\rho(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

sending t to the $\mathbb{C}_t$-twisted L^2-torsion.

Its value at $t = 1$ is just the L^2-torsion.

- On the analytic side this corresponds for closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_\pi \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

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Theorem (Lück)

Suppose that $\widetilde{X}$ is L^2-acyclic.

1. *The L^2 torsion function $\rho^{(2)} := \rho^{(2)}(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$ is well-defined.*
2. *The limits $\lim_{t\to\infty} \frac{\rho^{(2)}(t)}{\ln(t)}$ and $\lim_{t\to 0} \frac{\rho^{(2)}(t)}{\ln(t)}$ exist and we can define the degree of ϕ*

   $$\deg(X;\phi) \in \mathbb{R}$$

   to be their difference.
3. *There is a ϕ-twisted Fuglede-Kadison determinant*

   $$\det^{(2)}_{\mathrm{tw},\phi}\colon K_1^w(\mathbb{Z}G) \to \operatorname{map}((0,\infty),\mathbb{R})$$

   which sends $\rho_u^{(2)}(\widetilde{X})$ to $\rho^{(2)}(\widetilde{X};\phi)$.

Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_i(F) = \sum_{C \in \pi_0(M)} \min\{-\chi(C), 0\}.$$

- Thurston showed that this definition extends to the real vector space $H^1(M;\mathbb{R})$ and defines a seminorm on it.
- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle and $\phi = \pi_1(p)$, then

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Theorem (Friedl-Lück)

Let M be a 3-manifold. Then for every $\phi \in H^1(M; \mathbb{Z})$ we get the equality

$$\deg(M; \phi) = x_M(\phi).$$

Polytope homomorphism

- Consider the projection

$$\mathrm{pr}\colon G \to H_1(G)_f := H_1(G)/\operatorname{tors}(H_1(G)).$$

  Let K be its kernel.
- After a choice of a set-theoretic section of pr we get isomorphisms

$$\begin{aligned} \mathbb{Z}K * H_1(G)_f &\xrightarrow{\cong} \mathbb{Z}G; \\ S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big) &\xrightarrow{\cong} \mathcal{D}(G), \end{aligned}$$

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- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\text{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f), u_h \neq 0\}.$$

- The convex hull of $\text{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H^1(M;\mathbb{R}).$$

  This is the non-commutative analogon of the Newton polygon associated to a polynomial in several variables.

- Recall the Minkowski sum of two polytopes P and Q

$$P + Q = \{p + q \mid p \in P, q \in Q\}$$

  It satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$.

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Definition (Polytope group)

Let $\mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$ be the Grothendieck group of the abelian monoid of integral polytopes under the Minkowski sum modulo translations by elements in $H_1(G)_f$, where integral means that all extreme points lie on the lattice $H_1(G)_f$.

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) * H_1(G)_f$.
- Hence we can define a homomorphism of abelian groups

$$P' \colon \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f)\right)^{\times} \to \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

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- The composite

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times \xrightarrow{\cong} \Big(S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big)\Big)^\times \xrightarrow{P'} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

factories to the polytope homomorphism

$$P\colon \operatorname{Wh}^w(G) \to \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f).$$

Definition (Thurston polytope)

Let M be a 3-manifold. Define the Thurston polytope to be subset of $H^1(M;\mathbb{R})$

$$T(M) := \{\phi \in H^1(M;\mathbb{R}) \mid x_M(\phi) \leq 1\}.$$

Theorem (Friedl-Lück)

Let M be a 3-manifold. Then the image of the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$ under the polytope homomorphism

$$P\colon \operatorname{Wh}^w(\pi_1(M)) \to \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f)$$

is represented by the dual of the Thurston polytope, which is an integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f = H_1(M;\mathbb{R}) = H^1(M;\mathbb{R})^*$.

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Summary

- We can assign to a finite CW-complex X its universal L^2-torsion

$$\rho^{(2)}(\widetilde{X}) \in \mathrm{Wh}^w(\pi),$$

  provided that $\widetilde{X}$ is L^2-acyclic and π satisfies the Atiyah Conjecture.

- These assumptions are always satisfied for 3-manifolds.

- The Alexander polynomial is a special case.

- One can read of from the universal L^2-torsion a polytope which for a 3-manifold is the dual of the Thurston polytope.

- One can twist the L^2-torsion by a cycle $\phi \in H^1(M)$ and obtain a L^2-torsion function from which one can read of the Thurston norm.

L^2-Euler characteristic

Definition (L^2-Euler characteristic)

Let Y be a G-space. Suppose that

$$h^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0} b_n^{(2)}(Y;\mathcal{N}(G)) < \infty.$$

Then we define its L^2-Euler characteristic

$$\chi^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0}(-1)^n \cdot b_n^{(2)}(Y;\mathcal{N}(G)) \quad \in \mathbb{R}.$$

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Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.

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Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.

- Let $f\colon X \to X$ be a selfhomotopy equivalence of a connected finite CW-complex. Let T_f be its mapping torus. The projection $T_f \to S^1$ induces an epimorphism $\phi\colon \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

  Then $\widetilde{T_f}$ is L^2-acyclic and we get

$$\chi^{(2)}(\widetilde{T_f}; \phi) = \chi(X).$$

Theorem (Friedl-Lück)

Let M be a 3-manifold and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

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Higher order Alexander polynomials

- Higher order Alexander polynomials were introduced for a covering $G \to \overline{M} \to M$ of a 3-manifold by Harvey and Cochrane, provided that G occurs in the rational derived series of $\pi_1(M)$.
- At least the degree of these polynomials is a well-defined invariant of M and G.
- We can identify the degree with the ϕ-L^2-Euler characteristic.
- Thus we can extend this notion of degree also to the universal covering of M and can prove the conjecture that the degree coincides with the Thurston norm.

Group automorphisms

Theorem (Lück)

Let $f\colon X \to X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2-Betti numbers $b_n^{(2)}(\widetilde{T_f})$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f\colon G \to G$ be a group automorphism of the group G. Suppose that there is a finite model for BG and G satisfies the Atiyah Conjecture. Then we can define the universal L^2-torsion of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \mathrm{Wh}^w(G \rtimes_f \mathbb{Z})$$

- This seems to be a very powerful invariant which needs to be investigated further.

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- This seems to be a very powerful invariant which needs to be investigated further.

- It has nice properties, e.g., it depends only on the conjugacy class of f, satisfies a sum formula and a formula for exact sequences.
- If G is amenable, it vanishes.
- If G is the fundamental group of a compact surface F and f comes from an automorphism $a\colon F \to F$, then T_f is a 3-manifold and a lot of the material above applies.
- For instance, if a is irreducible, $\rho_u^{(2)}(f)$ detects whether a is pseudo-Anosov since we can read off the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition of T_f.

- Suppose that $H_1(f) = \mathrm{id}$. Then there is an obvious projection

$$\mathrm{pr}\colon H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \to H_1(G)_f.$$

Let

$$P(f) \in \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

be the image of $\rho_u^{(2)}(f)$ under the composite

$$\mathrm{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}(\mathrm{pr})} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

- What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where F is a finitely generated free group, is of particular interest.

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Summary (continued)

- The Thurston norm can also be read of from an L^2-Euler characteristic.
- The higher order Alexander polynomials due to Harvey and Cochrane are special cases of the universal L^2-torsion and we can prove the conjecture that their degree is the Thurston norm.
- The universal L^2-torsion seems to give an interesting invariant for elements in $\mathrm{Out}(F_n)$ and mapping class groups.