Asymptotics of the Period Map

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June 19, 2012
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Hodge Decomposition

Hodge Decomposition

Let X be a smooth, n-dimensional, compact Kähler manifold. Then $H^k(X, \mathbb{C})$ decomposes as:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q} \; ; \quad H^{q,p} = \overline{H^{p,q}},$$

where $H^{p,q}$ may be described as the set of cohomology classes admitting a representative of bidegree (p, q).

Requires existence of Kähler structure ω but depends only on complex structure.

Lefschetz Theorems and Primitive Cohomology

Lefschetz Theorem

If $k = n - \ell$, $\ell \geq 0$, then multiplication by powers ω^j of a Kähler class is injective for $j \leq \ell$ and an isomorphism for $j = \ell$.

The *primitive cohomology* is then defined as

$$H^k_0(X) := \{\alpha \in H^k(X) \, : \, \omega^{\ell+1} \cup \alpha = 0\}$$

The Hodge structure restricted to the primitive cohomology is polarized; i.e. satisfies the Hodge-Riemann bilinear relations:

Hodge-Riemann Bilinear Relations

Define a real bilinear form Q on $H^*(X, \mathbb{C})$ by

$$Q(\alpha, \beta) = (-1)^{\frac{k(k-1)}{2}} \int_X \alpha \cup \beta \, ,$$

where $\deg(\alpha) = k$ and the right-hand side is assumed to be zero if $\deg(\alpha \cup \beta) \neq 2 \dim_{\mathbb{C}}(X)$.

Then, the Hodge decomposition on $H^k(X, \mathbb{C})$ is Q-orthogonal (first bilinear relation) and:

$$i^{p-q} \, Q(\alpha, \omega^\ell \cup \bar{\alpha}) \geq 0 \quad \text{(second bilinear relation)}$$

for any

$$\alpha \in H^{p,q}(X) \cap H_0^{p+q}(X, \mathbb{C}) \, ; \quad k = p + q = n - \ell \, .$$

Moreover, equality holds if and only if $\alpha = 0$.

Families

Varying the complex structure of X defines a variation of the Hodge structure in cohomology.

Typically this arises from a family of varieties:

$$f\colon \mathcal{X} \subset \mathbb{P}^N \to S,$$

where f is a proper holomorphic submersion so that the fibers $X_b = f^{-1}(b)$ are smooth projective varieties.

Example: $\mathcal{X} = \{(x, y, t) : y^2 = x(x-1)(x-t)\}$;
$S = \{t : 0 < |t| < 1\}$.
The fibers X_t are curves of genus 1.

Monodromy

Such a family is locally C^∞-trivial. Globally, the diffeomorphism between fibers is only well defined up to homotopy. At the cohomology level we get a homomorphism:

$$\rho\colon \pi_1(S, s_0) \to \mathrm{Aut}_{\mathbb{Z}}(H^k(X_{s_0}, \mathbb{Z}), Q_k) =: G_{\mathbb{Z}}$$

called the *monodromy representation*.

Example: If $S = (\Delta^*)^r$ then $\pi_1(S) \cong \mathbb{Z}^r$ and ρ is determined by r commuting elements:

$$\gamma_1, \ldots, \gamma_r \in G_{\mathbb{Z}}.$$

Theorem: (Landman; Katz) The γ_i are quasi-unipotent; i.e. $\gamma_i^{u_i} = e^{N_i}$. Moreover, $N_i^{k+1} = 0$.

Classifying Space

We fix the data $(V_{\mathbb{Z}}, k, h^{p,q}, Q)$ and consider the space D of all Q-polarized Hodge structures on $V_{\mathbb{C}}$, with these invariants.

To a Hodge decomposition we associate a filtration

$$F^p := \bigoplus_{a \geq p} H^{a,k-a} \; ; \quad F^p \oplus \overline{F^{k-p+1}} = V_{\mathbb{C}}.$$

Conversely such a filtration defines a decomposition:

$$H^{p,q} := F^p \cap \overline{F^q}.$$

$$D^{\mathrm{open}} \subset \{\text{flags} \cdots F^p \subset F^{p-1} \subset \cdots : Q(F^p, F^{k-p+1}) = 0\} =: \check{D}$$

$\check{D}$ is a smooth projective variety. The group $G_{\mathbb{C}} := \mathrm{Aut}(V_{\mathbb{C}}, Q)$ acts transitively on $\check{D}$ and D is an orbit of $G_{\mathbb{R}} := \mathrm{Aut}(V_{\mathbb{R}}, Q)$:

$$\check{D} \cong G_{\mathbb{C}}/B \; ; \quad D \cong G_{\mathbb{R}}/V.$$

Example: weight 2

$$V_{\mathbb{C}} = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \,;\; H^{0,2} = \overline{H^{2,0}}$$

The polarization Q is symmetric, negative-definite on the real subspace $(H^{2,0} \oplus H^{0,2}) \cap V_{\mathbb{R}}$ and positive-definite on $H^{1,1} \cap V_{\mathbb{R}}$.

$$G_{\mathbb{R}} \cong O(2h^{2,0}, h^{1,1}) \,; \quad V \cong U(h^{2,0}) \times O(h^{1,1}).$$

$$\check{D} = \{F^2 \subset F^1 : Q(F^2, F^1) = 0\} \subset \mathcal{G}(h^{2,0}, V_{\mathbb{C}}) \times \mathcal{G}(h^{2,0} + h^{1,1}, V_{\mathbb{C}}).$$

Period Map

A family $f\colon \mathcal{X} \to S$ defines a map

$$\Phi\colon S \to G_{\mathbb{Z}} \backslash D$$

The local liftings to D are holomorphic and satisfy differential equations: Griffiths' transversality (aka: Horizontality).

The tangent space to D is a subspace of the tangent space to the product of Grasmannians:

$$T_F D \subset \bigoplus_p Hom(F^p, V_{\mathbb{C}}/F^p).$$

Griffiths' Transversality

Griffiths' Transversality Theorem

The differential of the period map takes values on the subspace

$$\bigoplus_p Hom(F^p, F^{p-1}/F^p).$$

Example: If $\Phi \colon U \subset \mathbb{C} \to D$ is a period map of PHS of weight two. Then the subspace $F(z) := F^2(z)$ determines the Hodge filtration and:

$$Q(F(z), F'(z)) = 0.$$

Abstract Variations of Hodge Structure

A VHS of weight k consists of: a local system $\mathcal{V}_{\mathbb{Z}} \to S$ and a filtration of the associated holomorphic vector bundle $\mathbb{V}$:

$$\cdots \subset \mathbb{F}^p \subset \mathbb{F}^{p-1} \subset \cdots \subset \mathbb{V}$$

by holomorphic subbundles such that:

- $\mathbb{V} = \mathbb{F}^p \oplus \overline{\mathbb{F}^{k-p+1}}$.
- $\nabla(\mathcal{F}^p) \subset \Omega^1_S \otimes \mathcal{F}^{p-1}$, where ∇ is the flat connection on $\mathbb{V}$.

The VHS is polarized if there exists a flat, non-degenerate, bilinear form $\mathcal{Q}$ defined over $\mathbb{Z}$, which polarizes the HS on each fiber.

Hodge Structure in the Lie Algebra

If $V_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ is a *PHS* on V, we get a HS of weight 0 on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus \mathfrak{g}^{r,-r} \; ; \quad [\mathfrak{g}^{r,-r}, \mathfrak{g}^{s,-s}] \subset \mathfrak{g}^{r+s,-r-s}$$

where

$$\mathfrak{g}^{r,-r} := \{X : X(H^{p,q}) \subset H^{p+r,q-r}\}.$$

We have $\mathfrak{b} = \mathrm{Lie}(B) = F^0\mathfrak{g} = \bigoplus_{r\geq 0} \mathfrak{g}^{r,-r}$. The adjoint action of B leaves $F^{-1}\mathfrak{g} = \bigoplus_{r\geq -1} \mathfrak{g}^{r,-r}$. invariant. The corresponding homogeneous vector bundle on $G_{\mathbb{C}}/B$ is the horizontal subbundle.

Example

Let $V_{\mathbb{C}} = \mathbb{C}^2 = H^{1,0} \oplus H^{0,1} = \mathbb{C} \cdot \begin{pmatrix} i \\ 1 \end{pmatrix} \oplus \mathbb{C} \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix}$

Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{s}l(2, \mathbb{C}) = \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{-1,1}$$

$$= \mathbb{C} \cdot \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \oplus \mathbb{C} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \oplus \mathbb{C} \cdot \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

Asymptotic Behavior of the Period Map

Consider a period map

$$\Phi\colon (\Delta^*)^r \to G_{\mathbb{Z}}\backslash D$$

with (unipotent) monodromy $\gamma_j = e^{N_j},\ j = 1, \ldots, r$, where

$$N_j \in \mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G_{\mathbb{R}}).$$

The map $\Psi(t_1, \ldots, t_r) = \exp\left(-\sum_j \frac{\log t_j}{2\pi i} N_j\right) \cdot \Phi(t_1, \ldots, t_r)$

is univalued with values in $\check{D}$.

Schmid's Nilpotent Orbit Theorem

Nilpotent Orbit Theorem

- *The map* Ψ *extends holomorphically to* Δ^r.
- *For* $|t| < \varepsilon$*, the map*

$$(t_1, \ldots, t_r) \mapsto \exp\left(\sum_j \frac{\log t_j}{2\pi i} N_j\right) \cdot F_{\lim}\,; \;\; F_{\lim} := \Psi(0) \in \check{D}$$

is an (abstract) period map, called a nilpotent orbit.
(Horizontality $\Leftrightarrow$ $N_j(F^p_{\lim}) \subset F^{p-1}_{\lim}$*.)*

- *The nilpotent orbit approximates the period map* Φ *exponentially.*

Reformulation of Schmid's Nilpotent Orbit Theorem

Let $\mathcal{V} \to (\Delta^*)^r$ be a VPHS with unipotent monodromy. Then the vector bundle $\mathbb{V}$ has an extension

$$\overline{\mathbb{V}} \to \Delta^r$$

whose sections around $0 \in \Delta^r$ are of the form

$$\tilde{v}(t) = \exp\left(\sum_{j=1}^{r} \frac{\log t_j}{2\pi i} N_j\right) \cdot v(t)$$

where $v(t)$ is a (multivalued) flat section of $\mathbb{V}$. Then, the Nilpotent Orbit Theorem asserts that the Hodge bundles $\mathbb{F}^p$ extend to $\overline{\mathbb{V}}$.

Mixed Hodge Structures

A (real) mixed Hodge structure on V consists of:

- An increasing filtration $W = \{W_\ell\}$ defined over $\mathbb{R}$, and
- A decreasing filtration $F = \{F^p\}$, such that

F induces a Hodge structure of weight ℓ on Gr_ℓ^W; i.e. the filtration

$$F^p(Gr_\ell^W) := (F^p \cap W_\ell + W_{\ell-1})/W_{\ell-1}$$

is a Hodge structure of weight ℓ.

Mixed Hodge Structures and Bigradings

Theorem

(Deligne) There is an equivalence between mixed Hodge structures (W, F) *on* V *and bigradings*

$$V_{\mathbb{C}} = \bigoplus_{a,b} I^{a,b}$$

such that

$$I^{a,b} \equiv \overline{I^{b,a}} \left(mod \bigoplus_{r<a,\, s<b} I^{r,s} \right).$$

If equality holds then we say that (W, F) *is split over* $\mathbb{R}$*.*

Given the bigrading we set:

$$W_\ell = \bigoplus_{a+b\leq\ell} I^{a,b}\,; \quad F^p = \bigoplus_{a\geq p} I^{a,b}.$$

Weight Filtration

Given $N\colon V \to V$, nilpotent, we can define a *unique increasing filtration* $W_\ell(N)$ such that:

- $N(W_\ell(N)) \subset W_{\ell-2}(N)$, and
- For $\ell \geq 0$, $N^\ell \colon \mathrm{Gr}^W_\ell := W_\ell / W_{\ell-1} \to \mathrm{Gr}^W_{-\ell}$ is an isomorphism.

Example: If $N^2 = 0$, then

$$\{0\} \subset W_{-1} = \mathrm{Im}(N) \subset W_0 = \ker(N) \subset W_1 = V.$$

$$N\colon Gr^W_1 = V/\ker(N) \xrightarrow{\cong} \mathrm{Im}(N) = Gr^W_{-1}.$$

Limiting MHS and Nilpotent Orbits

Theorem (Griffiths, Deligne, Schmid, Kaplan, C.)

Suppose

$$\exp\left(\sum_{j=1}^{r} \frac{\log t_j}{2\pi i} N_j\right) \cdot F\,; \quad F \in \check{D},$$

is a nilpotent orbit of Q-polarized HS of weight k. Then

1. $N(F^p) \subset F^{p-1}$ *and* $N^{k+1} = 0$ *for every N in the open cone* $C := \{\sum_j \lambda_j N_j, \lambda_j \in \mathbb{R}_{>0}\}$.
2. *For every* $N, N' \in C$, $W(N) = W(N') := W(C)$
3. $(W(C)[-k], F)$ *is a MHS (The limiting MHS.)*
4. *For every* $N \in C$, *the form* $Q(\bullet, N^\ell \bullet)$ *polarizes the HS (of weight* $k+\ell$ *in the subspace:*

$$\mathrm{Pr}_{k+\ell} := ker\{N^{\ell+1} : Gr^W_{k+\ell} \to Gr^W_{k-\ell-2}\}\,; \quad W = W(C)[-k].$$

Limiting MHS and Nilpotent Orbits

Theorem (Griffiths, Deligne, Schmid, Kaplan, C.)

Conversely, if $(N_1, \ldots, N_r; F, Q, k)$, $N_j \in \mathfrak{g}_{\mathbb{R}}$, $F \in \check{D}$, *satisfy the above four properties, then the map*

$$(t_1, \ldots, t_r) \to \exp\left(\sum_{j=1}^{r} \frac{\log t_j}{2\pi i} N_j\right) \cdot F$$

is a VPHS for $|t| < \varepsilon$.

SL_2-orbits and Polarized Split MHS

Suppose (W, F_0), $W = W(N)[-k]$, is a polarized MHS split over $\mathbb{R}$. Then

$$V_{\mathbb{R}} = \bigoplus_{\ell=0}^{2k} V_\ell \; ; \quad (V_\ell)_{\mathbb{C}} = \bigoplus_{p+q=\ell} I^{p,q} \; ; \quad I^{q,p} = \overline{I^{p,q}}.$$

We let Y denote the real linear transformation defined by $Y(v) = (\ell - k) \cdot v$ if $v \in V_\ell$. Note that $Y \in \mathfrak{g}_{\mathbb{R}}$ and

$$N(I^{p,q}) \subset I^{p-1,q-1} \Rightarrow [Y, N] = -2N.$$

SL_2-orbits and Polarized Split MHS

Theorem

If $(W(N)[-k], F_0)$ is a PMHS split over $\mathbb{R}$, then there exist real representations

$$\rho_* \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}_\mathbb{C} \; ; \quad \rho \colon SL_2(\mathbb{C}) \to G_\mathbb{C}$$

such that

$$\rho_* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = Y \; ; \quad \rho_* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N$$

giving rise to a horizontal equivariant morphism $\tilde{\rho} \colon \mathbb{P}^1 \to \check{D}$; $\tilde{\rho}(g \cdot i) = \rho(g) \cdot (e^{iN}.F_0)$ mapping the upper-half plane $U \subset \mathbb{P}^1$ to $D \subset \check{D}$. Moreover, ρ_* is a morphism of Hodge structures (induced by $i \in U$ and $\exp(iN) \in D$). (Hodge representation).

SL_2-orbits and Polarized Split MHS

Conversely, any such Hodge representation arises from a PMHS split over $\mathbb{R}$.

There are canonical constructions that associate a split MHS to a PMHS. One such construction is given by Schmid's SL_2-orbit Theorem which also provides a detailed description of the relationship with the nilpotent orbit.

A different functorial construction is due to Deligne.

These results extend to several variables.

MHS in the Lie Algebra

Suppose $(W(C)[-k], F, Q, k)$ is a PMHS on $V_{\mathbb{C}}$. Let $\{I^{p,q}\}$ denote the associated bigrading. Then, we can define a bigrading of $\mathfrak{g}_{\mathbb{C}}$ by

$$I^{a,b}\mathfrak{g} := \{X \in \mathfrak{g}_{\mathbb{C}} : X(I^{p,q}) \subset I^{p+a,q+b}\}.$$

This defines a MHS. Moreover $[I^{a,b}\mathfrak{g}, I^{a',b'}\mathfrak{g}] \subset I^{a+a',b+b'}\mathfrak{g}$.

We have: $\mathrm{Lie}(\mathrm{Stab}(F)) = F^0\mathfrak{g} = \bigoplus_{a\geq 0} I^{a,b}\mathfrak{g}$ and

$$\mathfrak{g}_- := \bigoplus_{a<0} I^{a,b}\mathfrak{g}$$

is a complementary subspace. So we have a local model for $\check{D}$ near F:

$$X \in \mathfrak{g}_- \mapsto \exp(X) \cdot F \in \check{D}.$$

Asymptotics

Let $\Phi\colon (\Delta^*)^r \to G_{\mathbb{Z}}\backslash D$ be a period mapping with monodromy $N_1, \ldots, N_r$ and $F_{\lim}$ the limiting Hodge Filtration. Then

$$\Psi(t) = \exp \Gamma(t) \cdot F_{\lim}, \quad \text{and}$$

$$\Phi(t) = \exp \left(\sum_j \frac{\log t_j}{2\pi i} N_j \right) \cdot \exp \Gamma(t) \cdot F_{\lim},$$

where $\Gamma\colon \Delta^r \to \mathfrak{g}_-$ is holomorphic and $\Gamma(0) = 0$. We can write

$$\Gamma(t) = \sum_{a<0} \Gamma_a(t) \; ; \quad \Gamma_a(t) \in \bigoplus_b I^{a,b}\mathfrak{g}$$

Asymptotics

Theorem

If Φ *is horizontal then*

$$X(t) = \sum_j \frac{\log t_j}{2\pi i} N_j + \Gamma_{-1}(t)$$

satisfies the differential equation

$$dX \wedge dX = 0.$$

Conversely, if $Y \colon \Delta^r \to \bigoplus_b I^{-1,b}\mathfrak{g}$, $Y(0) = 0$, *is holomorphic and*

$$X(t) = \sum_j \frac{\log t_j}{2\pi i} N_j + Y(t)$$

satisfies the differential equation then, there exists a unique period map with $\Gamma_{-1}(t) = Y(t)$.

Weight-three Example (Deligne)

Consider a PVHS over Δ^* of weight 3 and $h^{ij} = 1$. Assume that the limiting MHS splits over $\mathbb{R}$ and $N^3 \neq 0$. Then

$$V_{\mathbb{C}} = I^{0,0} \oplus I^{1,1} \oplus I^{2,2} \oplus I^{3,3}$$

and

$$\Gamma_{-1}(t) \;=\; \begin{pmatrix} 0 & a(t) & 0 & 0 \\ 0 & 0 & b(t) & 0 \\ 0 & 0 & 0 & a(t) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Weight-three Example (Deligne)

The definition of $\Gamma(t)$ depends on the choice of parameter t. One can show that for $q := t\exp(2\pi i a(t))$, $a(q) = 0$. This parameter is canonical.

Thus, the period map depends only on the nilpotent orbit and one analytic function $b(q)$.

The function $b(q)$ also has a nice interpretation: For a suitably normalized choice $\omega(q) \in H^{3,0}(q) = F^3(q)$, we may define

$$\kappa := Q\left(\omega(q), \Theta^3(\omega(q))\right) \; ; \quad \Theta := 2\pi i q \frac{d}{dq}.$$

This is called the *normalized Yukawa coupling*. We have:

$$\kappa = 2\pi i q \frac{db}{dq}.$$