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TOEPLITZ ALGEBRAS ASSOCIATED TO ISOMETRIC FLOWS

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Introduction

Let M be a compact Riemannian manifold, and let $\Phi = \{\phi_t\}$ be a smooth one-parameter group of isometries of M. The group Φ is called an *isometric flow* on M. In this paper, we associate a Toeplitz C^*-algebra $\mathcal{T}(\Phi)$ to an isometric flow Φ on M, and begin to study how the C^*-algebraic properties of $\mathcal{T}(\Phi)$ are related to the geometric and topological properties of Φ.

The algebra $\mathcal{T}(\Phi)$ is defined as follows. Since each map ϕ_t is an isometry of M, it induces a unitary operator U_t on $L^2(M)$, where M is endowed with the usual measure coming from the Riemannian metric. Let D be the infinitesimal generator of the group $\{U_t\}$, and let P be the positive spectral projection of D. Then $\mathcal{T}(\Phi)$ is the C^*-subalgebra of $\mathcal{L}(L^2(M))$ generated by the set $\{PM_fP\colon f \in C(M)\}$, where M_f is pointwise multiplication by f.

Toeplitz C^*-algebras have been used by many researchers to study a variety of problems in geometry and analysis, and Toeplitz algebras have been particularly important in the study of flows. In addition, the C^*-algebras one obtains by this construction tend to be very interesting from an operator-algebraic point of view. Indeed, the Toeplitz algebra on the circle, i.e., the C^*-algebra generated by the unilateral shift, is precisely the Toeplitz algebra one gets by taking M to be the unit circle in the standard metric and where $\Phi = \{\phi_t\}$ is defined by letting ϕ_t be a rotation of $2\pi t$ for each t in $\mathbb{R}$.

There have been several papers written about Toeplitz operators and Toeplitz algebras associated to flows. Toeplitz algebras for irrational flows on tori were considered in [JX] and [JK]. These researchers examined the dependence of the Toeplitz algebra on the particular irrational flow chosen, and they also computed the K-theory of the Toeplitz algebras and their commutator ideals. It is natural to generalize the situation studied in [JX] and [JK] to topological flows on compact Hausdorff spaces, and this has been done in a series of papers. In [CMX], the spectral theory and index theory of Toeplitz operators were explored. In [MPX], the authors studied the Toeplitz algebra associated to a strictly ergodic flow, and they computed the K-theory of the Toeplitz algebra and some related algebras. In [MX], Toeplitz algebras for $\mathbb{R}^n$-actions were defined, and the results obtained specialize in the case $n = 1$ to give improved results

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to those in [MPX]. In particular, this latter paper allows the authors to replace the hypotheses of minimality and strict ergodicity of the flow with a weaker condition.

In a different direction, in [Pa] we considered Toeplitz algebras for flows defined by one-parameter subgroups of compact Lie groups. These flows are typically neither minimal nor strictly ergodic, and the methods used in [Pa] have more of a topological flavor than the techniques used in the other papers mentioned above.

The current paper concerns itself with a much more general class of flows than in [Pa]. We require our flows to act smoothly, so in one sense our setup is less general than the one in [CMX], [MPX], and [MX]. However, by working in the smooth category, we do not have to make any other restrictions on the flows we consider. For example, we allow the flow to have fixed points.

Isometric flows have a wide range of behaviors. At one extreme, they may be minimal, as in [JX] and [JK]. On the other hand, it is possible that all the orbits of an isometric flow are closed, and hence very far from being minimal. In addition, as we mentioned above, Φ may have fixed points. We show that the closures of orbits of Φ partition M into compact submanifolds that are either tori or single points. Furthermore, when Φ is restricted to one of these submanifolds, then one either has a point with the trivial flow on it, or an irrational flow on a torus. We prove that the Toeplitz algebras over the orbit closures of Φ form a continuous field of C^*-algebras, and that $\mathcal{T}(\Phi)$ is isomorphic to the C^*-algebra of continuous sections of this field. From this, we deduce that $\mathcal{T}(\Phi)$ is a diffeomorphism invariant of the flow Φ; i.e., if Φ is an isometric flow of M for two different metrics, then the corresponding Toeplitz algebras are isomorphic.

This paper is organized as follows. In Section 1, we collect the various geometric properties of isometric flows that we need. Several of the results in this section may be known to geometers, but the author could find no reference for them, so proofs of these results are included. In Section 2, we precisely define the Toeplitz algebra $\mathcal{T}(\Phi)$, and we define the aforementioned continuous field of Toeplitz algebras. We also define a measurable field of Hilbert spaces, and use this measurable field to prove that $\mathcal{T}(\Phi)$ is isomorphic to the C^*-algebra of sections of the continuous field of Toeplitz algebras. Finally, we show that $\mathcal{T}(\Phi)$ only depends on the smooth properties of Φ.

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1. Properties of isometric flows

In this section, we collect the geometric properties of isometric flows that we will use in the rest of the paper.

PROPOSITION 1.1. *The orbit closures of Φ form a partition of M.*

Proof. Let $\sim$ denote the relation of being in the same orbit closure of Φ. Clearly $\sim$ is both reflexive and symmetric. To see that $\sim$ is transitive, note that $x \sim y$

means that there exists a sequence $\{s_n\}$ of real numbers such that $\lim_{n\to\infty} \phi_{s_n}(x) = y$. With this in mind, suppose that $x \sim y$ and $y \sim z$. Then $\lim_{n\to\infty} \phi_{s_n}(x) = y$ and $\lim_{n\to\infty} \phi_{t_n}(y) = z$ for some sequences $\{s_n\}$ and $\{t_n\}$. Therefore $\lim_{n\to\infty} \phi_{t_n+s_n}(x) = \lim_{n\to\infty} \phi_{t_n}(\phi_{s_n}(x)) = \lim_{n\to\infty} \phi_{t_n}(y) = z$. □

The next result on the orbit closures of Φ is well known to many geometers; the proof given here is adapted from the remarks at the beginning of Section 2 in [Ca2].

PROPOSITION 1.2. *Let p be a point in M, and let $\Phi(p)$ denote the orbit of p. Then the closure $\overline{\Phi(p)}$ of $\Phi(p)$ is a submanifold of M that is isometric to a torus $\mathbb{T}^k$ with a bi-invariant metric. Furthermore, under this isometry, Φ restricted to $\overline{\Phi(p)}$ conjugates to (multiplication by) a dense one-parameter subgroup of $\mathbb{T}^k$.*

Proof. Note that Φ is a subgroup of the compact Lie group $Iso(M)$ of isometries of M, and therefore $\overline{\Phi}$ is a Lie subgroup of $Iso(M)$. Since Φ is an abelian group and a connected topological space, $\overline{\Phi}$ is also abelian and connected, and is therefore isomorphic to a torus. By Corollary VI.1.4 in [Br], $\overline{\Phi}(p)$ is diffeomorphic to the quotient of $\overline{\Phi}$ by the stabilizer subgroup $\overline{\Phi}_p$ of the point p, and thus $\overline{\Phi}(p)$ is diffeomorphic to a torus $\mathbb{T}^k$. This implies that $\overline{\Phi}(p)$ is closed in M, and therefore $\overline{\Phi}(p) = \overline{\Phi(p)}$. Use the diffeomorphism between $\overline{\Phi(p)}$ and $\overline{\Phi}/\overline{\Phi}_p$ to make $\overline{\Phi(p)}$ into a group. Then it is clear that $\Phi(p)$ is a dense one-parameter subgroup of $\overline{\Phi(p)}$. In addition, multiplication by elements of Φ preserves the metric on $\overline{\Phi(p)}$, and since these elements are dense in $\overline{\Phi(p)}$, the metric on $\overline{\Phi(p)}$ is bi-invariant. □

Let N be a closed submanifold of M. Then it is known (see [Mo], Section 3.1, for example) that for some $\epsilon > 0$, the set $V_\epsilon = \{p \in M: d(p, N) \leq \epsilon\}$ has the following properties:

(i) For each point $x \in V_\epsilon$, there exists a unique point y in N closest to x, and these points are connected by a unique geodesic that realizes this minimal distance.
(ii) The projection from V_ϵ to N given by mapping each point of V_ϵ to the closest point of N defines a locally trivial fibration whose fibers are balls in $\mathbb{R}^k$, where k is the codimension of N in M.

Definitions. We call V_ϵ the *closed tubular neighborhood* of radius ϵ about N. We call the boundary C_ϵ of V_ϵ the *cylinder* of radius ϵ about N.

PROPOSITION 1.3. *Let N be a connected proper submanifold of M, and suppose that N is a union of orbit closures of Φ. Then there exists a positive number ϵ with the property that for each positive number $\alpha \leq \epsilon$, the cylinder C_α of radius α about N is a union of orbit closures of Φ. Furthermore, there exists a diffeomorphism $\psi_\alpha: C_\epsilon \to C_\alpha$ that intertwines the action of Φ on C_ϵ and C_α. As a consequence, ψ_α conjugates the orbit closures of Φ in C_ϵ to the orbit closures of Φ in C_α.*

Proof. Choose ϵ so small that N has a tubular neighborhood V of radius ϵ. Since Φ preserves distance and maps N to itself, for any positive real number α less than ϵ, the cylinder C_α about N is a union of orbits. Since C_α is a closed set, it is also the union of orbit closures.

Let x be a point in C_ϵ, and let p be the point in N closest to x. Let γ be the geodesic joining x and p, and define $\psi_\alpha(x)$ to be the unique point on γ that is in C_α and is between x and p. Then the properties of tubular neighborhoods guarantee that ψ_α is a diffeomorphism. To see that ψ_α intertwines Φ, take ϕ_t in Φ. Since ϕ_t preserves distances, we necessarily have that $\phi_t(p)$ is the point in N closest to $\phi_t(x)$. Moreover, since ϕ_t maps geodesics to geodesics, $\phi_t(\gamma)$ is the geodesic through $\phi_t(p)$ and $\phi_t(x)$, and thus $\psi_\alpha(\phi_t(x)) = \phi_t(\psi_\alpha(x))$. The last statement in the theorem then follows immediately. □

Remark 1.4. Note that Proposition 1.3 also implies that a tubular neighborhood V_ϵ of N is a union of orbit closures.

Remark 1.5. When N is an orbit closure that is not a single point, then Proposition 1.3 follows directly from Theorem II.C.3 in [Ca1]. However, since we allow Φ to have fixed points, a more general result that Carrière's is necessary. Also, we will need the full strength of Proposition 1.3 in future papers.

PROPOSITION 1.6. *Let B be the quotient space obtained by identifying points of M that are in the same orbit closure of Φ. Then B is a compact Hausdorff space.*

Proof. Let π denote the quotient map. Since M is compact, $\pi(M) = B$ is compact as well. To show that the Hausdorff condition is satisfied, let b_1 and b_2 be distinct points in B. Proposition 1.3 and the Remark 1.4 guarantee that we can find neighborhoods V_1 and V_2 of b_1 and b_2, respectively, that are unions of orbit closures. In addition, it is clear that we can choose V_1 and V_2 so small that they do not intersect. Thus $\pi(V_1)$ and $\pi(V_2)$ are nonintersecting neighborhoods of b_1 and b_2, respectively. □

2. The Toeplitz algebra associated to an isometric flow

In this section, we take an isometric flow Φ on a compact Riemannian manifold M, define a Toeplitz C^*-algebra $\mathcal{T}(\Phi)$, and study the structure of this C^*-algebra. It will be convenient to adopt the following notational convention: given f in $C(M)$ and b in B, f^b will denote the restriction of f to the set $\pi^{-1}(b)$.

Equip M with the measure μ determined by the Riemannian metric. Each isometry ϕ_t in Φ induces a unitary operator U_t on $L^2(M)$ via the formula $U_t(\xi) = \xi \circ \phi_t$, and so we have a one-parameter group $\{U_t\}$ of unitaries on $L^2(M)$. Let D be the infinitesimal generator of this group. Then D is a self-adjoint operator on $L^2(M)$

whose domain includes the smooth functions on M. Use the functional calculus for self-adjoint operators to define $P = I - \chi_{(-\infty,0)}(D)$, where $\chi_{(-\infty,0)}$ denotes the characteristic function of the negative real numbers. Note that typically P is just the positive spectral projection of D. However, this more complicated definition of P is necessary for dealing with situations where Φ has fixed points; see the comments following Proposition 2.3. Finally, for each continuous complex-valued function f on M, define a bounded operator M_f on $L^2(M)$ via multiplication: $M_f(\xi) = f\xi$.

Definition. The Toeplitz algebra of Φ, denoted $\mathcal{T}(\Phi)$, is the C^*-subalgebra of $\mathcal{L}(L^2(M))$ generated by $\{PM_fP\colon\ f \in C(M)\}$.

The goal of this section is to describe $\mathcal{T}(\Phi)$ in terms of Toeplitz algebras on the orbit closures of Φ. For each point b in B, restrict the metric on M to get a metric on $\pi^{-1}(b)$. Equip $\pi^{-1}(b)$ with the measure μ^b that this metric induces, and let $\operatorname{vol}(\pi^{-1}(b))$ be the volume of $\pi^{-1}(b)$ in the induced metric. If $\pi^{-1}(b)$ is a single point, we give it the counting measure, so in this case $\operatorname{vol}(\pi^{-1}(b)) = 1$.

PROPOSITION 2.1. *The function $\Psi\colon M \to \mathbb{R}$ given by $\Psi(p) = \frac{1}{\operatorname{vol}(\pi^{-1}(\pi(p)))}$ is in $L^1(M)$.*

Proof. We induct on the dimension of the manifold M. If M is one-dimensional, then the result is easily seen to be true. Now let M be an n-dimensional manifold, and suppose that the proposition is true for manifolds of dimension less than n. For each orbit closure N of Φ, choose a small tubular neighborhood of N, as in the proof of Proposition 1.3. Since M is compact, a finite number of these tubes cover M, and it clearly suffices to show that the integral of Ψ is finite when restricted to the closure of each of the tubes. Therefore, let $\overline{V}$ be the closed tube about an orbit closure N. Since $\overline{V}$ is compact, we can modify the metric on $\overline{V}$ without affecting the finiteness of the integral of Ψ over $\overline{V}$. Now, $\overline{V}$ is diffeomorphic to $N \times \mathbb{B}^k$ for some integer k, where $\mathbb{B}^k$ denotes the closed unit ball in $\mathbb{R}^k$. Endow N with the metric it inherits from M, give $\mathbb{B}^k$ its standard metric, and equip $\overline{V}$ with the product metric. Since $\overline{V}$ is a tubular neighborhood of N, we may choose the diffeomorphism $\overline{V} \cong N \times \mathbb{B}^k$ so that cylinders are mapped to cylinders and geodesics perpendicular to N are mapped to geodesics perpendicular to N. Then

$$\int_{\overline{V}} \Psi \, d\operatorname{vol}_{\overline{V}} = \int_{N\times\mathbb{B}^k} \Psi \, d\operatorname{vol}_N \, d\operatorname{vol}_{\mathbb{B}^k} = \int_0^1 \int_{N\times rS^{k-1}} \Psi \, d\operatorname{vol}_N \, d\sigma r^{k-1} dr,$$

where $d\sigma$ denotes the standard volume form on the unit $(k-1)$-sphere S^{k-1}, and where rS^{k-1} is the $(k-1)$-sphere of radius r. Let L be an orbit closure in $N \times S^{k-1}$, and for each $0 < r \leq 1$, let $\psi_r\colon N \times S^{k-1} \to N \times rS^{k-1}$ be the diffeomorphism defined in Proposition 1.3. Then $\operatorname{vol}(\psi_r(L)) \geq r^{k-1}\operatorname{vol}(L)$, and so $\Psi(x, r\sigma) \leq$

$r^{-(k-1)}\Psi(x,\sigma)$, where $x \in N$ and $\sigma \in S^{k-1}$. Therefore

$$\begin{aligned}\int_{\overline{V}} \Psi \, d\,\mathrm{vol}_{\overline{V}} &= \int_0^1 \int_{N\times rS^{k-1}} \Psi \, d\,\mathrm{vol}_N \, d\sigma r^{k-1} dr \\ &\leq \int_0^1 \int_{N\times S^{k-1}} r^{-(k-1)}\Psi \, d\,\mathrm{vol}_N \, d\sigma r^{k-1} dr \\ &= \int_0^1 \left[\int_{N\times S^{k-1}} \Psi \, d\,\mathrm{vol}_N \, d\sigma\right] dr.\end{aligned}$$

By our inductive hypothesis, the integral of Ψ over $N \times S^{k-1}$ is finite, and therefore the integral of Ψ over $\overline{V}$ is finite. $\square$

Define ρ: $C(B) \to \mathbb{C}$ by $\rho(g) = \int_M \pi^*(g)\Psi \, d\mu$, where π^*: $C(B) \to C(M)$ is the map induced from π: $M \to B$. In light of Proposition 2.1, ρ is a bounded linear functional on $C(B)$. Since ρ is obviously positive, it determines a Borel measure ν on B, and we have $\rho(g) = \int_M \pi^*(g)\Psi \, d\mu = \int_B g \, d\nu$ for all g in $C(B)$.

For each f in $C(M)$, define λ_f in $\prod_{b\in B} L^2(\pi^{-1}(b))$ by $\lambda_f(b) = f^b$, and let Γ be the set

$$\left\{\eta\colon B \to \prod_{b\in B} L^2(\pi^{-1}(b)) \;\middle|\; b \mapsto \langle \eta(b), \lambda_f(b)\rangle \text{ is } \nu\text{-measurable for all } f \in C(M)\right\}.$$

By Lemma 8.10 in [Ta], $(\prod_{b\in B} L^2(\pi^{-1}(b)), \Gamma)$ is a measurable field of Hilbert spaces over B.

LEMMA 2.2. *Let f be in $C(M)$. There exists a continuous function h on M with the following properties*:

(i) *h is constant on the orbit closures of Φ*;
(ii) $\|f\|_2 = \|h\|_2$;
(iii) $\|f^b\|_2 = \|h^b\|_2$ *for all $b \in B$*.

Proof. Let d denote the geodesic distance function on M. Fix p_0 in M, and choose an increasing sequence $\{t_n\}$ of positive real numbers such that $d(p_0, \phi_{t_n}(p_0)) < \frac{1}{n}$ for each natural number n and such that $\lim_{n\to\infty} t_n = \infty$. For each n, let $T_n = (\lceil t_n \rceil)^2$, where $\lceil t_n \rceil$ is the smallest integer greater than or equal to t_n. Let $F = f\bar{f}$, and define a function F_n by the formula

$$F_n(x) = \frac{1}{T_n + 1} \sum_{j=0}^{T_n} F\left(\phi_{\frac{jt_n}{T_n}}(x)\right).$$

Since F is a positive function, it is evident from the definition that $\|F_n\|_1 = \|f\bar{f}\|_1 = \|f\|_2$ and that $\|F_n^b\|_1 = \|(f\bar{f})^b\|_1 = \|f^b\|_2$ for all b in B. Clearly the collection

$\{F_n\}$ is pointwise bounded, and a simple argument using the triangle inequality and the uniform continuity of F shows that $\{F_n\}$ is equicontinuous. By the Arzelà-Ascoli theorem, there exists a subsequence $\{F_{n_k}\}$ that converges uniformly to a continuous function H_1; I claim that H_1 is constant on the set $\overline{\Phi(p_0)}$.

Note that every point x in $\overline{\Phi(p_0)}$ is arbitrarily close to a point of the form $\phi_{\frac{mt_{n_k}}{T_{n_k}}}(x)$, where $0 \leq m \leq T_{n_k}$. Therefore, to prove the claim, it suffices to show that given $\epsilon > 0$, we can choose k so large that $|F_{n_k}(\phi_{\frac{mt_{n_k}}{T_{n_k}}}(p_0)) - F_{n_k}(p_0)| < \epsilon$ for all $0 \leq m \leq T_{n_k}$. Now,

$$\begin{aligned}\left|F_{n_k}(\phi_{\frac{mt_{n_k}}{T_{n_k}}}(p_0)) - F_{n_k}(p_0)\right| &= \frac{1}{T_{n_k}+1}\left|\sum_{j=m}^{T_{n_k}} F(\phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0)) + \sum_{j=T_{n_k}+1}^{T_{n_k}+m} F(\phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0))\right.\\ &\qquad \left. - \sum_{j=0}^{m-1} F(\phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0)) - \sum_{j=m}^{T_{n_k}} F(\phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0))\right|.\end{aligned}$$

Cancelling the first and fourth terms and reindexing the second term, we get

$$\begin{aligned}\left|F_{n_k}(\phi_{\frac{mt_{n_k}}{T_{n_k}}}(p_0)) - F_{n_k}(p_0)\right| &= \frac{1}{T_{n_k}+1}\left|\sum_{j=1}^{m} F(\phi_{t_{n_k}+\frac{jt_{n_k}}{T_{n_k}}}(p_0)) - \sum_{j=0}^{m-1} F(\phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0))\right|\\ &\leq \frac{1}{T_{n_k}+1}\left[\left|F(\phi_{t_{n_k}+\frac{mt_{n_k}}{T_{n_k}}}(p_0)) - F(p_0)\right|\right.\\ &\qquad \left. + \left|\sum_{j=1}^{m-1} F(\phi_{t_{n_k}+\frac{jt_{n_k}}{T_{n_k}}}(p_0)) - F(\phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0))\right|\right].\end{aligned}$$

Choose δ so that $d(y, z) < \delta$ implies $|F(y) - F(z)| < \frac{\epsilon}{2}$ for all y and z in M. Next, choose k so large that $\frac{2\|F\|_\infty}{T_{n_k}+1} < \frac{\epsilon}{2}$ and $d(p_0, \phi_{t_{n_k}}(p_0)) < \delta$. The latter requirement implies that $d(\phi_{t_{n_k}+\frac{jt_{n_k}}{T_{n_k}}}(p_0), \phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0)) < \delta$ for each j, and hence $|F(\phi_{t_{n_k}+\frac{jt_{n_k}}{T_{n_k}}}(p_0)) - F(\phi_{\frac{jt_{n_k}}{T_{n_k}}}(p_0))| < \frac{\epsilon}{2}$. Therefore, we have

$$\begin{aligned}\left|F_{n_k}(\phi_{\frac{mt_{n_k}}{T_{n_k}}}(p_0)) - F_{n_k}(p_0)\right| &< \frac{1}{T_{n_k}+1}\left[\left|F(\phi_{t_{n_k}+\frac{mt_{n_k}}{T_{n_k}}}(p_0)) - F(p_0)\right| + (m-1)\frac{\epsilon}{2}\right]\\ &\leq \frac{1}{T_{n_k}+1}\left[2\|F\|_\infty + (m-1)\frac{\epsilon}{2}\right]\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

and thus H_1 is constant on $\overline{\Phi(p_0)}$.

Choose a dense sequence $\{p_n\}$ of points in M. At the beginning of this proof, we constructed the function H_1 from the initial function F and point p_0; replace F by H_1 and p_0 by p_1, and repeat this construction to obtain a function H_2. Continue this process; that is, replace F by H_n and p_0 by p_n to obtain a function H_{n+1}. The sequence $\{H_n\}$ is pointwise bounded and equicontinuous, so we may again apply Arzelà-Ascoli to show that there is a subsequence $\{H_{n_k}\}$ that converges uniformly to a continuous function H. Let $h = \sqrt{H}$; I claim that h possesses the desired properties.

To show that h is constant on each orbit closure of Φ, it of course suffices to show that H is constant on the orbit closures. Fix $\epsilon > 0$, and choose δ so that $d(y, z) < \delta$ implies $|H(y) - H(z)| < \frac{\epsilon}{2}$ for all points y and z in M. Now choose y and z to be arbitrary points of M that are in the same orbit closure of Φ, and choose a natural number l so that $d(y, p_l) < \delta$. Since y and z are in the same orbit closure, we can choose a point q_l in $\overline{\Phi(p_l)}$ with $d(z, q_l) < \delta$. Then

$$\begin{aligned}|H(y) - H(z)| &\leq |H(y) - H(p_l)| + |H(p_l) - H(q_l)| + |H(q_l) - H(z)| \\ &< \frac{\epsilon}{2} + |H(p_l) - H(q_l)| + \frac{\epsilon}{2}.\end{aligned}$$

But $|H(p_l) - H(q_l)| = \lim_{k\to\infty} |H_{n_k}(p_l) - H_{n_k}(q_l)|$, and since $H_{n_k}(p_l) = H_{n_k}(q_l)$ for $n_k > l$, we have $|H(y) - H(z)| < \epsilon$, and hence H is constant on orbit closures. To see that (ii) holds, observe that H is the norm limit of functions $\{H_n\}$ with the same L^1 norm as F, and so $\|f\|_2 = \|F\|_1 = \|H\|_1 = \|h\|_2$. Similarly, $\|f^b\|_2 = \|F^b\|_1 = \|H^b\|_1 = \|h^b\|_2$. □

PROPOSITION 2.3. *Let $\int_B^\oplus L^2(\pi^{-1}(b))\, d\nu$ be the direct integral of the measurable field $(\prod_{b\in B} L^2(\pi^{-1}(b)), \Gamma)$, and define $\Lambda\colon C(M) \to \int_B^\oplus L^2(\pi^{-1}(b))\, d\nu$ by $\Lambda(f) = \lambda_f$. Then Λ extends to a unitary operator from $L^2(M)$ to $\int_B^\oplus L^2(\pi^{-1}(b))\, d\nu$.*

Proof. Since Λ maps $C(M)$ onto a dense subset of $\int_B^\oplus L^2(\pi^{-1}(b))\, d\nu$, it suffices to show that $\|f\|_2 = \|\lambda_f\|_2$ for all f in $C(M)$. Fix $f \in C(M)$, let h be the function constructed in Lemma 2.2, and choose g in $C(B)$ so that $h = \pi^*(g)$. Then from Lemma 2.2 and the definitions of Ψ and ν, we have the following string of equalities:

$$\begin{aligned}\|f\|_2^2 &= \|h\|_2^2 = \int_M |h(p)|^2\, d\mu \\ &= \int_M |h(p)|^2 \mathrm{vol}(\pi^{-1}(\pi(p))\Psi\, d\mu \\ &= \int_M \pi^*\left(|g|^2 \mathrm{vol}(\pi^{-1}(\cdot))\right) \Psi\, d\mu \\ &= \int_B |g(b)|^2 \mathrm{vol}(\pi^{-1}(b))\, d\nu\end{aligned}$$

$$
\begin{aligned}
&= \int_B \int_{\pi^{-1}(b)} |h|^2 \, d\mu^b \, d\nu \\
&= \int_B \|h^b\|_2^2 \, d\nu \\
&= \int_B \|f^b\|_2^2 \, d\nu \\
&= \int_B \|\lambda_f(b)\|_2^2 \, d\nu \\
&= \|\lambda_f\|_2^2.
\end{aligned}
$$

□

We now look at Toeplitz algebras associated to the orbit closures of Φ. For each b in B, Φ acts on $\pi^{-1}(b)$, and the restriction of Φ to $\pi^{-1}(b)$ defines a one-parameter group of unitaries on $L^2(\pi^{-1}(b))$. Let D^b be the infinitesimal generator of this group, and let $P^b = I - \chi_{(-\infty,0)}(D^b)$. Note that if $\pi^{-1}(b)$ is a single point, then $D^b = 0$, so $P^b = I - \chi_{(-\infty,0)}(0) = I$.

Definition. Let $b \in B$. The Toeplitz algebra of Φ restricted to $\pi^{-1}(b)$, denoted $\mathcal{T}^b(\Phi)$, is the C^*-subalgebra of $\mathcal{L}(L^2(\pi^{-1}(b)))$ generated by $\{P^b M_g P^b\colon g \in C(\pi^{-1}(b))\}$, where M_g is the operator of multiplication by g on $L^2(\pi^{-1}(b))$.

For each f in $C(M)$, let $\theta_f \in \prod_{b\in B} \mathcal{T}^b(\Phi)$ be defined as $\theta_f(b) = P^b M_{f^b} P^b$, and let Θ be the collection of all finite linear combinations and products of the θ_f's.

PROPOSITION 2.4. *For each θ in Θ, the map from B to $[0,\infty)$ given by $b \mapsto \|\theta(b)\|_\infty$ is continuous.*

Proof. It suffices to show that for each f in $C(M)$, the map $b \mapsto \|\theta_f(b)\|_\infty$ is continuous. Note that $\|\theta_f(b)\|_\infty = \|P^b M_{f^b} P^b\|_\infty \le \|P^b\|_\infty \|M_{f^b}\|_\infty \|P^b\|_\infty = \|f^b\|_\infty$. On the other hand, the map $P^b M_{f^b} P^b \mapsto f^b$ extends to a surjective homomorphism $\sigma\colon \mathcal{T}^b(\Phi) \to C(\pi^{-1}(b))$ (trivially true when $\pi^{-1}(b)$ is a single point, and true by [JK] otherwise), whence $\|\theta_f(b)\|_\infty = \|P^b M_{f^b} P^b\|_\infty \ge \|\sigma(P^b M_{f^b} P^b)\|_\infty = \|f^b\|_\infty$. Therefore $\|\theta_f(b)\|_\infty = \|f^b\|_\infty$, and the map $b \mapsto \|f^b\|_\infty$ is obviously continuous. □

From Proposition 2.4, we see that $(\prod_{b\in B} \mathcal{T}^b(\Phi), \overline{\Theta})$ is a continuous field of C^*-algebras over B, where $\overline{\Theta}$ is the collection of continuous vector fields generated by Θ (see [Di], Propositions 10.2.3 and 10.3.2).

THEOREM 2.5. *The map $PM_fP \mapsto \theta_f$ extends to a C^*-algebra isomorphism from $\mathcal{T}(\Phi)$ to the C^*-algebra of sections of $(\prod_{b\in B} \mathcal{T}^b(\Phi), \overline{\Theta})$.*

Proof. Let $W: L^2(M) \to \int_B^\oplus L^2(\pi^{-1}(b))\,d\nu$ be the unitary operator constructed in Proposition 2.3. Then $WM_fW^* = \int_B^\oplus M_{f^b}\,d\nu$ for every $f \in C(M)$. In addition, W intertwines the action of Φ on M with its action on the orbit closures of Φ, and hence $WDW^* = \int_B^\oplus D^b\,d\nu$ and $WPW^* = \int_B^\oplus P^b\,d\nu$. Therefore $WPM_fPW^* = \int_B^\oplus P^bM_{f^b}P^b\,d\nu$, and W implements the desired isomorphism. □

We can now state and prove the main result of this paper.

THEOREM 2.6. *Let Φ be an isometric flow on a compact manifold M. Then the Toeplitz algebra $\mathcal{T}(\Phi)$ does not depend upon the Riemannian metric. That is, if Φ is an isometric flow under two metrics g_1 and g_2, then the corresponding Toeplitz algebras are isomorphic.*

Proof. Let Φ_1 and Φ_2 denote Φ with the metrics g_1 and g_2, respectively. To prove the theorem, it suffices to show that the continuous fields $(\prod_{b\in B}\mathcal{T}^b(\Phi_1), \overline{\Theta})$ and $(\prod_{b\in B}\mathcal{T}^b(\Phi_2), \overline{\Theta})$ are isomorphic. To this end, first note that if $\pi^{-1}(b)$ is a point, then $\mathcal{T}^b(\Phi_1) = \mathcal{T}^b(\Phi_2)$. Now suppose that $\pi^{-1}(b)$ is a torus, and let μ_1^b and μ_2^b be the measures on $\pi^{-1}(b)$ that are associated to g_1 and g_2, respectively. Then Proposition 1.2 implies that $\mu_2^b = c_b\mu_1^b$ for some constant c_b, and therefore multiplication by $\sqrt{c_b}$ defines a unitary operator from $L^2(\pi^{-1}(b), g_1)$ to $L^2(\pi^{-1}(b), g_2)$; this unitary operator induces an isomorphism between $\mathcal{T}^b(\Phi_1)$ and $\mathcal{T}^b(\Phi_2)$. In addition, this family of isomorphisms maps the set of sections Θ into itself, so by Proposition 10.2.4 in [Di], the continuous fields are isomorphic. □

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