**Abstract.** Let $n \geq 2$ be an integer. An $n$-potent is an element $e$ of a ring $R$ such that $e^n = e$. In this paper, we study $n$-potents in matrices over $R$ and use them to construct an abelian group $K_0^n(R)$. If $A$ is a complex algebra, there is a group isomorphism $K_0^n(A) \cong (K_0(A))^{n-1}$ for all $n \geq 2$. However, for algebras over cyclotomic fields, this is not true in general. We consider $K_0^n$ as a covariant functor, and show that it is also functorial for a generalization of homomorphism called an $n$-homomorphism.

1. **Introduction**

For more than thirty years, $K$-theory has been an essential tool in studying rings and algebras [1, 2]. Given a ring $R$, the simplest functorial object associated to $R$ is the abelian group $K_0(R)$. There are multiple ways of defining $K_0(R)$, but the most useful characterization when working with operator algebras is to define $K_0(R)$ in terms of idempotents (or projections, if an involution is present) in matrix algebras over $R$; i.e., elements $e$ in $M_k(R)$ for some $k$ with the feature that $e^2 = e$. In this paper, we define, for each natural number $n \geq 2$, a group which we denote $K_0^n(R)$. This group is constructed from matrices $e$ over $R$ with the property that $e^n = e$; we call such matrices $n$-potents. We define $K_0^n(R)$ for all rings, unital or not, and show that $K_0^n$ determines a covariant functor from rings to abelian groups.

Let $\mathbb{Q}(n-1)$ be the cyclotomic field obtained from the rationals by adjoining the $(n-1)$-th roots of unity. We show that $K_0^n$ is half-exact on the subcategory of $\mathbb{Q}(n-1)$-algebras, and given any such algebra $A$, we show that $K_0^n(A)$ is isomorphic to a direct sum of $n-1$ copies of $K_0(A)$. Since a $\mathbb{C}$-algebra $A$ is a $\mathbb{Q}(n-1)$-algebra for all $n$, whatever invariants are contained in $K_0^n(A)$ are already contained in $K_0(A)$. However, $K_0^p$ for $p \neq n$ may generate new groups for cyclotomic algebras, e.g., $K_0^4(\mathbb{Q}(4)) \cong \mathbb{Z} \oplus 2\mathbb{Z}$ (Theorem 2) which is not isomorphic to $K_0^3(\mathbb{Q}(3)) \cong \mathbb{Z}^3$. Thus, $K_0^4$ distinguishes between the fields $\mathbb{Q}(3)$ and $\mathbb{Q}(4)$, but idempotent, and also tripotent ($n = 3$), $K$-theory do not.

The paper is organized as follows. In Section 2, we define various notions of equivalence on the set of \( n \)-potents, and explore the relationships between these equivalence relations. Most of our results in this section mirror analogous facts about idempotents, but in many cases the proofs differ and/or are more difficult for \( n \)-potents. In Section 3, we define \( n \)-potent \( K \)-theory and study its properties and compute some examples. Finally, in Section 4, we consider \( n \)-homomorphisms on rings and algebras \([?, ?, ?]\), and show that \( n \)-potent \( K \)-theory is functorial for such maps; this is a phenomenon that does not appear in ordinary idempotent \( K \)-theory.

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Note: Unless stated otherwise, all rings and algebras have a unit; i.e., a multiplicative identity, and all ring and algebra homomorphisms are unital.

2. Equivalence of \( n \)-potents

In this section, we define \( n \)-potents, develop their basic theory, and discuss various equivalence relations on \( n \)-potents. We begin by looking at \( n \)-potents over general rings, but eventually we will specialize to get a well-behaved theory.

**Definition 2.1.** Let \( R \) be a ring and \( n \geq 2 \) a natural number. An element \( e \) in \( R \) is called an \( n \)-potent if \( e^n = e \). For \( n = 2, 3, 4 \), we use the terms idempotent, tripotent, and quadripotent, respectively. The set of all \( n \)-potents in \( R \) is denoted \( \mathcal{P}^n(R) \).

We begin with a very simple but useful fact about \( n \)-potents:

**Lemma 2.2.** Suppose \( e \) is an \( n \)-potent. Then \( e^{n-1} \) is an idempotent.

*Proof.* \((e^{n-1})^2 = e^{n-1}e^{n-1} = e^ne^{n-2} = ee^{n-2} = e^{n-1}\). \(\Box\)

**Definition 2.3.** Let \( e \) and \( f \) be \( n \)-potents in a ring \( R \). We say that \( e \) and \( f \) are algebraically equivalent and write \( e \sim_a f \) if there exist elements \( a \) and \( b \) in \( R \) such that \( e = ab \) and \( f = ba \). We say that \( e \) and \( f \) are similar and write \( e \sim_s f \) if there exists an invertible element \( z \) in \( R \) with the property that \( f = zez^{-1} \).

**Lemma 2.4.** Suppose that \( e \) and \( f \) are algebraically equivalent \( n \)-potents in a ring \( R \). Then the elements \( a \) and \( b \) described in Definition 2.3 can be chosen so that

\[
\begin{align*}
a &= e^{n-1}a = af^{n-1} = e^{n-1}af^{n-1} \\
b &= f^{n-1}b = be^{n-1} = f^{n-1}be^{n-1}.
\end{align*}
\]
Proof. Choose elements $\tilde{a}$ and $\tilde{b}$ in $R$ so that $\tilde{a}\tilde{b} = e$ and $\tilde{b}\tilde{a} = f$. Set $a = e^{n-1}\tilde{a}f^{n-1}$ and $b = f^{n-1}\tilde{b}e^{n-1}$. Using Lemma ??, we have

$$ab = (e^{n-1}\tilde{a}f^{n-1})(f^{n-1}\tilde{b}e^{n-1}) = e^{n-1}\tilde{a}f^{n-1}\tilde{b}e^{n-1} = e^{n-1}(\tilde{a}\tilde{b})e^{n-1} = e^{n-1}e = e^n = e.$$ 

Similarly, $ba = f$. The two strings of equalities in the statement of the lemma then follow easily. \hfill \square

Proposition 2.5. The relations $\sim_a$ and $\sim_s$ are equivalence relations on $P^n(R)$.

Proof. The only nonobvious point to establish is that $\sim_a$ is transitive. Let $e$, $f$, and $g$ be elements of $P^n(R)$, and suppose that $e \sim_a f \sim_a g$. Choose elements $a$, $b$, $c$ and $d$ in $R$ so that $e = ab$, $f = ba = cd$, and $g = dc$, and set $s = af^{n-2}c$ and $t = db$. Then

$$st = af^{n-2}cdb = af^{n-1}b = (ba)^{n-1}b = (ab)^n = e^n = e$$

and

$$ts = dbaf^{n-2}c = df^{n-1}c = d(cd)^{n-1}c = (dc)^n = g^n = g.$$ 

\hfill \square

Proposition 2.6. If $e$ and $f$ are similar $n$-potents in a ring $R$, then they are algebraically equivalent.

Proof. Choose a noninvertible element $z$ in $R$ such that $f = e^{-1}z$, and set $a = ez^{-1}$ and $b = ze^{n-1}$. Then $ab = e^n = e$ and $ba = ze^n z^{-1} = f$. \hfill \square

As is the case with idempotents, algebraic equivalence does not imply similarity in general. However, we do have the following result:

Proposition 2.7. Suppose that $e$ and $f$ are algebraically equivalent $n$-potents in a ring $R$. Then

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \sim_s \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$$

in the ring $M_2(R)$ of $2 \times 2$ matrices over $R$.

Proof. Choose elements $a$ and $b$ in $R$ so that $e = ab$ and $f = ba$; without loss of generality, we assume that $a$ and $b$ satisfy the conclusions of Lemma ???. Define

$$u = \begin{pmatrix} 1 - f^{n-1} & b \\ af^{n-2} & 1 - e^{n-1} \end{pmatrix}$$

and

$$v = \begin{pmatrix} 1 - e^{n-1} & e^{n-1} \\ e^{n-1} & 1 - e^{n-1} \end{pmatrix}.$$
Straightforward computation yields that both \( u^2 \) and \( v^2 \) equal the identity matrix in \( M_2(R) \), and thus each is its own inverse. Set \( z = uv \). Then
\[
z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{-1} = \begin{pmatrix} 1 - f^{n-2} & b \\ af^{n-2} & 1 - e^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 1 - f^{n-2} & b \\ af^{n-2} & 1 - e^{n-1} \end{pmatrix}
= \begin{pmatrix} beaf^{n-2} & 0 \\ 0 & 0 \end{pmatrix}.
\]
To complete the proof, we note that
\[
beaf^{n-2} = baba(ba)^{n-2} = (ba)^n = f^n = f.
\]
\[\square\]

**Definition 2.8.** We say \( n \)-potents \( e \) and \( f \) in a ring \( R \) are orthogonal if \( ef = fe = 0 \), in which case we write \( e \perp f \).

**Proposition 2.9.** Let \( e \) and \( f \) be orthogonal \( n \)-potents in a ring \( R \). Then \( (e + f)^k = e^k + f^k \). In particular, \( e + f \) is an \( n \)-potent.

**Proof.** We proceed by induction. Obviously the result holds for \( k = 1 \). Now suppose the result holds for an arbitrary natural number \( k \). Then
\[
(e + f)^{k+1} = (e + f)^{k}(e + f)
= (e^k + f^k)(e + f)
= e^{k+1} + e^k f + f^k e + f^{k+1}
= e^{k+1} + e^{k-1}(ef) + f^{k-1}(fe) + f^{k+1}
= e^{k+1} + f^{k+1}.
\]
\[\square\]

**Proposition 2.10.** For \( i = 1, 2 \), let \( e_i \) and \( f_i \) be algebraically equivalent \( n \)-potents in a ring \( R \). Suppose that \( e_1 \) and \( f_1 \) are orthogonal to \( e_2 \) and \( f_2 \), respectively. Then \( e_1 + e_2 \) and \( f_1 + f_2 \) are algebraically equivalent.

**Proof.** For \( i = 1, 2 \), choose \( a_i \) and \( b_i \) so that \( e_i = a_i b_i \), \( f_i = b_i a_i \), and so that \( a_i \) and \( b_i \) satisfy the conclusion of Lemma ??.
\[
a_1 b_2 = a_1 f_1^{n-1} f_2^{n-1} b_2 = 0.
\]
Similarly, \( b_2 a_1 \), \( a_2 b_1 \), and \( b_1 a_2 \) are also zero. Thus
\[
(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_2 b_2 = e_1 + e_2
\]
and
\[
(b_1 + b_2)(a_1 + a_2) = b_1 a_1 + b_2 a_2 = f_1 + f_2,
\]
whence \( e_1 + e_2 \) is algebraically equivalent to \( f_1 + f_2 \).
\[\square\]
Proposition 2.11. Let \( e \) and \( f \) be \( n \)-potents in a ring \( R \).

(a) \( \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \sim_a \begin{pmatrix} f & 0 \\ 0 & e \end{pmatrix} \) and \( \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \sim_a \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \).

(b) If \( e \perp f \) then \( \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \sim_a \begin{pmatrix} e + f & 0 \\ 0 & 0 \end{pmatrix} \).

Proof. Define

\[
\begin{aligned}
a &= \begin{pmatrix} 0 & e \\ f & 0 \end{pmatrix} \\
b &= \begin{pmatrix} 0 & f^{n-1} \\ e^{n-1} & 0 \end{pmatrix}
\end{aligned}
\]

in \( M_2(R) \). Then

\[
\begin{aligned}
ab &= \begin{pmatrix} 0 & e \\ f & 0 \end{pmatrix} \begin{pmatrix} 0 & f^{n-1} \\ e^{n-1} & 0 \end{pmatrix} = \begin{pmatrix} e^n & 0 \\ 0 & f^n \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \\
ba &= \begin{pmatrix} 0 & f^{n-1} \\ e^{n-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & e \\ f & 0 \end{pmatrix} = \begin{pmatrix} f^n & 0 \\ 0 & e^n \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & e \end{pmatrix},
\end{aligned}
\]

which establishes the first part of (a); to obtain the second part, simply take \( f \) to be zero.

To prove (b), first observe that if \( e \perp f \), then \( e + f \) is an \( n \)-potent by Proposition 2.11. Define

\[
\begin{aligned}
a &= \begin{pmatrix} e & 0 \\ f & 0 \end{pmatrix} \\
b &= \begin{pmatrix} e^{n-1} & f^{n-1} \\ 0 & 0 \end{pmatrix}
\end{aligned}
\]

Then

\[
\begin{aligned}
ab &= \begin{pmatrix} e & 0 \\ f & 0 \end{pmatrix} \begin{pmatrix} e^{n-1} & f^{n-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^n & ef^{n-1} \\ 0 & fn \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \\
ba &= \begin{pmatrix} e^{n-1} & f^{n-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} e^n + f^n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e + f & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}
\]

whence the result follows. \( \square \)

2.1. Cyclotomic Algebras. Fix an integer \( n \geq 2 \). Let \( \omega_0 = 0 \) and define \( \omega_k = e^{2\pi i (k-1)/(n-1)} \) for \( 1 \leq k \leq n-1 \). Note that \( \omega_1 = 1 \) and \( \omega_1, \ldots, \omega_{n-1} \) are the \( (n-1) \)-th roots of unity, with \( \omega_2 = \zeta_{n-1} = e^{2\pi i/(n-1)} \) a primitive \( (n-1) \)-th root of unity. Moreover, \( \Omega_n = \{ \omega_0, \omega_1, \ldots, \omega_{n-1} \} \) is the set of roots of the polynomial equation \( x^n - x = x^{n-1} - 1 = 0 \).

The cyclotomic field \( \mathbb{Q}(n-1) = \mathbb{Q}[\zeta_{n-1}] \) is obtained by adjoining the primitive \( (n-1) \)-th root of unity \( \zeta_{n-1} = \omega_2 \) to the field of rational numbers. This implies \( \Omega_n \subseteq \mathbb{Q}(n-1) \subseteq \mathbb{C} \) and so, in particular, every \( \mathbb{C} \)-algebra is canonically a \( \mathbb{Q}(n-1) \)-algebra for all \( n \).
Definition 2.12. Let $\mathbb{F}$ be a field and let $A$ be an $\mathbb{F}$-algebra with unit. An $n$-partition of unity is an ordered $n$-tuple $(e_0, e_1, \ldots, e_{n-1})$ of idempotents in $A$ such that

1. $e_0 + e_1 + \cdots + e_{n-1} = 1$;
2. $e_0, e_1, \ldots, e_{n-1}$ are pairwise orthogonal; i.e., $e_j e_k = \delta_{jk}1$ for all $0 \leq j, k \leq n - 1$.

Note that $e_0 = 1 - (e_1 + \cdots + e_{n-1})$ is completely determined by $e_1, e_2, \ldots, e_{n-1}$ and is thus redundant in the notation for an $n$-partition of unity. For $\mathbb{Q}(n - 1)$-algebras, we can write down a general expression for each $n$-potent:

Theorem 2.13. Let $A$ be a $\mathbb{Q}(n - 1)$-algebra with unit, and suppose $e$ is an $n$-potent in $A$. Then there exists a unique $n$-partition of unity $(e_0, e_1, \ldots, e_{n-1})$ in $A$ such that

$$e = \sum_{k=1}^{n-1} \omega_k e_k.$$ 

Proof. Let $p_0, p_1, \ldots, p_{n-1} \in \mathbb{Q}(n - 1)[x]$ be the Lagrange polynomials

$$p_k(x) = \frac{\prod_{j \neq k}(x - \omega_j)}{\prod_{j \neq k}(\omega_k - \omega_j)}.$$ 

In particular, $p_0(x) = 1 - x^{n-1}$. Each polynomial $p_k$ has degree $n - 1$ and satisfies $p_k(\omega_k) = 1$ and $p_k(\omega_j) = 0$ for all $j \neq k$. We claim that for all numbers $x \in \mathbb{Q}(n - 1) \subseteq \mathbb{C}$,

1. $$\sum_{k=0}^{n-1} p_k(x) = p_0(x) + \cdots + p_{n-1}(x) = 1$$

and that

2. $$x = \sum_{k=0}^{n-1} \omega_k p_k(x).$$

Indeed, these identities follow from the fact that these polynomial equations have degree $n - 1$ but are satisfied by the $n$ distinct points in $\Omega_n$.

Now, given any $x^n = x$ in $\Omega_n$ it follows that $p_k(x)^2 = p_k(x)$. Hence, for any $n$-potent $e \in A$, if we define $e_k = p_k(e)$, then each $e_k$ is an idempotent in $A$, and Equation (2) implies that

$$\sum_{k=0}^{n-1} e_k = \sum_{k=0}^{n-1} p_k(e) = 1.$$
These idempotents are pairwise orthogonal, because
\[ e_j e_k = p_j(e)p_k(e) = 0 \]
for \( j \neq k \). Finally,
\[ e = \sum_{k=1}^{n-1} \omega_k p_k(e) = \sum_{k=1}^{n-1} \omega_k e_k \]
by Equation (??). \(\square\)

2.2. Local Banach algebras. Although the \( n \)-potent results in this subsection and the next are not really necessary for \( n \)-potent \( K \)-theory (see Corollary ??), we include them for completeness and because some of their proofs are different from and/or more complicated than the corresponding facts about idempotents.

**Definition 2.14.** A normed \( \mathbb{C} \)-algebra \( A \) is called a local Banach algebra if for each natural number \( n \), the matrix algebra \( M_n(A) \) is closed under the holomorphic functional calculus.

**Proposition 2.15.** If \( e \) is a non-zero \( n \)-potent in a local Banach algebra \( A \), then \( \|e\| \geq 1 \).

**Proof.** We have \( 0 < \|e\| = \|e^n\| \leq \|e\|^n \), and so \( 1 \leq \|e\|^{n-1} \), which in turn implies \( \|e\| \geq 1 \). \(\square\)

Using the norm topology on local Banach algebras, we can define homtopy for \( n \)-potents:

**Definition 2.16.** Let \( e \) and \( f \) be \( n \)-potents in a local Banach algebra \( A \). We say that \( e \) and \( f \) are homotopic if there exists a norm-continuous path \( t \mapsto e_t \) of \( n \)-potents in \( A \) such that \( e_0 = e \) and \( e_1 = f \). If \( e \) and \( f \) are homotopic, we write \( e \sim_h f \).

**Lemma 2.17.** Let \( e \) and \( f \) be \( n \)-potents in a local Banach algebra. Then for all natural numbers \( j \geq 1 \),
\[ f^j - e^j = \sum_{k=1}^{j} f^{j-k}(f - e)e^{k-1}. \]
Proof. Obviously the result is true for \( j = 1 \). Suppose the equation holds for arbitrary \( j \). Then

\[
\begin{align*}
f^{j+1} - e^{j+1} & = f^{j+1} - fe^j + fe^j - e^{j+1} \\
& = f(f^j - e^j) + (f - e)e^j \\
& = \sum_{k=1}^{j} f^{j-k}(f - e)e^{k-1} + (f - e)e^j \\
& = \sum_{k=1}^{j+1} f^{(j+1)-k}(f - e)e^{k-1}.
\end{align*}
\]

\[ \square \]

**Theorem 2.18.** Let \( e \) and \( f \) be \( n \)-potents in a local Banach algebra \( A \). Set \( M = (1 + \| e \|)^{2n} \geq 1 \), and suppose that

\[
\| e - f \| < \frac{2(n - 1)}{(n^2 + n + 2)M} < 1.
\]

Then there exists an invertible element \( z \) in \( A \) such that \( \| z - 1 \| < 1 \) and \( zez^{-1} = f \). In particular, \( e \sim_s f \).

Proof. When \( n = 2 \), this fact is well known (see for example Proposition 4.3.2 in [?]), so we assume for the rest of the proof that \( n > 2 \). Define

\[
v = e^{n-1}f^{n-1} + (n - 1)(1 - e^{n-1})(1 - f^{n-1}) + \sum_{k=1}^{n-2} e^k f^{(n-1)-k}.
\]

Observing that \( e(1 - e^{n-1}) = 0 = (1 - f^{n-1})f \), we have

\[
ev = ef^{n-1} + \sum_{k=1}^{n-2} e^{k+1} f^{(n-1)-k}
\]

\[ = ef^{n-1} + \sum_{k=1}^{n-3} e^{k+1} f^{(n-1)-k} + e^{n-1} f \]

\[ = e^{n-1} f + \sum_{k=0}^{n-3} e^{k+1} f^{(n-1)-k} \]

\[ = e^{n-1} f + \sum_{k=1}^{n-2} e^k f^{n-k} \]

\[ = vf. \]
Thus, $ev = vf$. Next,

$$v - (n - 1) = \sum_{k=1}^{n-2} e^k f^{(n-1)-k} - (n - 2)e^{n-1}$$

$$+ (e^{n-1}f^{n-1} - e^{n-1})$$

$$+ (n - 1)(e^{n-1}f^{n-1} - f^{n-1})$$

$$= \sum_{k=1}^{n-2} (e^k f^{(n-1)-k} - e^k e^{(n-1)-k})$$

$$+ (e^{n-1}f^{n-1} - e^{n-1})$$

$$+ (n - 1)(e^{n-1}f^{n-1} - f^{n-1})$$

$$= \sum_{k=1}^{n-2} e^k (f^{(n-1)-k} - e^{(n-1)-k})$$

$$+ (e^{n-1}f^{n-1} - e^{n-2}f + e^{n-2}f - e^{n-1})$$

$$+ (n - 1)(e^{n-1}f^{n-1} - ef^{n-2} + ef^{n-2} - f^{n-1})$$.

Applying Lemma ?? to the first sum, we obtain

$$v - (n - 1) = \sum_{k=1}^{n-2} e^k \left( \sum_{j=1}^{(n-1)-k} f^{(n-1)-k-j} (f - e) e^{j-1} \right)$$

$$+ \left( e^{n-2}(e - f) f^{n-1} + e^{n-2}(f - e) \right)$$

$$+ (n - 1) \left( e^{n-1}(f - e) f^{n-2} + (e - f) f^{n-2} \right)$$.

The triangle inequality and the fact that $e$ and $f$ are $n$-potents give us the inequalities

$$\|e^j\| \|f^k\| \leq \|e\|^n \|f\|^n \leq (1 + \|e\|)^n (1 + \|e\|)^n = M$$

for all $j$ and $k$. Thus

$$\|v - (n - 1)\| \leq M\|e - f\| \sum_{k=1}^{n-2} \left( \sum_{j=1}^{(n-1)-k} (f - e) e^{j-1} \right)$$

$$+ 2M\|e - f\|$$

$$+ 2(n - 1)M\|e - f\|$$

$$= M\|e - f\| \sum_{k=1}^{n-2} \left( (n - 1) - k \right)$$

$$+ 2nM\|e - f\|$$
\[
M \left( \frac{(n-1)(n-2)}{2} \right) \| e - f \|
+ 2nM \| e - f \|
= M \left( \frac{n^2 + n + 2}{2} \right) \| e - f \|
< M \left( \frac{n^2 + n + 2}{2} \right) \left( \frac{2(n-1)}{M(n^2 + n + 2)} \right)
= n - 1.
\]

Therefore \( \| v - (n-1) \| < n - 1 \) and hence \( \| v/(n-1) - 1 \| < 1 \). Thus \( v/(n-1) \) is invertible and implements the desired similarity. \( \square \)

If we make the convention in the previous proof that all summands whose upper index is \( n - 2 \) are set to zero when \( n = 2 \), then we obtain an alternate proof of Theorem ?? for idempotents.

A simple compactness and continuity argument on the interval \([0, 1]\) as in Proposition 4.3.3 of [?]? yields the following corollary:

**Corollary 2.19.** Let \( \{ e_t \} \) be a homotopy of \( n \)-potents in a local Banach algebra \( A \). Then \( e_0 \) is similar to \( e_1 \).

**Proposition 2.20.** Suppose that \( e \) and \( f \) are similar \( n \)-potents in a local Banach algebra \( A \). Then

\[
\begin{pmatrix}
e & 0 \\
0 & 0
\end{pmatrix}
\sim_h
\begin{pmatrix}
f & 0 \\
0 & 0
\end{pmatrix}.
\]

**Proof.** Choose \( z \) in \( A \) so that \( zez^{-1} = f \). By Proposition 3.4.1 [?] there is a norm-continuous path \( R_t \) of invertibles in \( M_2(A) \) from

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
to \begin{pmatrix}
z & 0 \\
0 & z^{-1}
\end{pmatrix}.
\]

Then

\[
e_t = R_t \begin{pmatrix}
e & 0 \\
0 & 0
\end{pmatrix} R_t^{-1}
\]

is the desired homotopy. \( \square \)

2.3. **Local \( C^* \)-algebras.** We investigate the relationship between similarity and unitary equivalence, and show that every \( n \)-potent is similar to a normal \( n \)-potent in a local \( C^* \)-algebra.

**Definition 2.21.** A local \( C^* \)-algebra is a local Banach \( * \)-algebra \( A \) satisfying \( \| x^*x \| = \|x\|^2 \) for all \( x \) in \( A \).
Recall that an element \( x \) in a local \( C^* \)-algebra \( A \) is called normal if \( x^*x = xx^* \). It is well known ([?], Proposition 4.6.2) that every idempotent is similar to a projection \((p^2 = p^* = p)\) and, by a spectral argument, normal idempotents are projections. Our goal is to show that every \( n \)-potent is similar to a normal \( n \)-potent and that similarity and unitary equivalence are the same for normal \( n \)-potents in a local \( C^* \)-algebra. We begin with a definition.

**Definition 2.22.** An \( n \)-partition of unity \((e_0, e_1, \ldots, e_{n-1})\) in a local \( C^* \)-algebra \( A \) is self-adjoint if \( e_k = e_k^* \) for each \( 0 \leq k \leq n - 1 \); i.e., if each \( e_k \) is a projection. A partition of unity \((f_0, f_1, \ldots, f_{n-1})\) is similar to \((e_0, e_1, \ldots, e_{n-1})\) if there exists an invertible element \( z \) in \( A \) with the property that \( ze_kz^{-1} = f_k \) for all \( k \).

**Lemma 2.23.** Let \( A \) be a local \( C^* \)-algebra. The following statements are equivalent:

(a) \( e \) is a normal \( n \)-potent in \( A \);
(b) \( e \) is a partial isometry \((e = ee^*e)\) in \( A \) and \( e^* = e^{n-2} \);
(c) the \( n \)-partition of unity \((e_0, \ldots, e_{n-1})\) associated to an \( n \)-potent element \( e \) in \( A \) is self-adjoint.

**Proof.** Clearly (b) implies (a), because \( e^n = ee^{n-2}e = ee^*e = e \) and \( ee^* = ee^{n-2} = e^{n-1} = e^*e \). Now, suppose \( e \) is a normal \( n \)-potent. Then every complex number \( \lambda \) in the spectrum of \( e \) satisfies the equation \( \lambda^n = \lambda \), whence \( \lambda = 0 \) or \( \lambda^{n-1} = 1 = \lambda \bar{\lambda} \). In either case, \( \bar{\lambda} = \lambda^{n-2} \), and the functional calculus implies that \( e^* = e^{n-1} \). Thus \( e = e^n = ee^{n-2}e = ee^*e \), i.e., \( e \) is a partial isometry, and so (a) implies (b).

Now, if \( e \) is a normal \( n \)-potent and we write \( e = \sum_{k=1}^{n-1} \omega_k e_k \), then each \( e_k \) is a projection. For if \( e \) commutes with its adjoint, the same is true of each \( e_k \), and self-adjoint idempotents are projections by the spectral theorem. Thus, (a) implies (c). The fact that (c) implies (a) is trivial. \( \square \)

**Proposition 2.24.** Every \( n \)-partition of unity in a local \( C^* \)-algebra \( A \) is similar to a self-adjoint \( n \)-partition of unity.

**Proof.** We induct on \( n \). For \( n = 2 \), Proposition 4.6.2 in [?] immediately yields that there exists an invertible element \( z \) and a normal idempotent \((i.e., a projection) p_0 \) such that \( ze_0z^{-1} = p_0 \). Then

\[
p_1 = ze_1z^{-1} = z(1 - e_0)z^{-1} = 1 - p_0
\]

is also a projection, and thus \((e_0, e_1)\) is similar to \((p_0, p_1)\).

Now, assume the result is true for \( n \) and for all local \( C^* \)-algebras. Take an \((n + 1)\)-partition of unity \((e_0, e_1, \ldots, e_n)\) in \( A \), and let \( u \) be an
invertible in $A$ that conjugates the idempotent $e_n$ to a projection $p_n$ in $A$. Let $p = 1 - p_n$. Then

$$u(e_0 + \cdots + e_{n-1})u^{-1} = u(1 - e_n)u^{-1} = 1 - p_n = p.$$  

Let $f_k = ue_ku^{-1}$ for $0 \leq k \leq n - 1$. Then $(f_0, \ldots, f_{n-1})$ is an $n$-partition of unity in the unital local $C^*$-algebra $pAp$. By the induction hypothesis there is an invertible $w$ in $pAp$ that conjugates $(f_0, \ldots, f_{n-1})$ to a self-adjoint $n$-partition of unity $(p_0, \ldots, p_{n-1})$. Note that because $w$ is invertible in $pAp$, we have $ww^{-1} = w^{-1}w = p$, as well as $wp = pw = w$. Hence, $p_nw = wp_n = w(1 - p) = 0$ and $w^{-1}p_n = p_nw^{-1} = 0$. Furthermore,

$$p_0 + \cdots + p_{n-1} = p = 1 - p_n,$$

and so $(p_0, \ldots, p_n)$ is a self-adjoint $(n + 1)$-partition of unity for $A$.

We claim that $(e_0, \ldots, e_n)$ is similar to $(p_0, \ldots, p_n)$. Indeed, if we set $z = (w + p_n)u$ then

$$zu^{-1}(w^{-1} + p_n) = (w + p_n)(w^{-1} + p_n) = ww^{-1} + p_n = p + p_n = 1.$$  

Similarly $zu^{-1}(w^{-1} + p_n) = 1$, and so $z$ is invertible. For $0 \leq k \leq n - 1$,

$$ze_kz^{-1} = (w + p_n)f_k(w^{-1} + p_n) = wf_kw^{-1} = p_k,$$

because $f_k$ is perpendicular to $1 - p = p_n$. Finally,

$$ze_nz^{-1} = (w + p_n)p_n(w^{-1} + p_n) = p_n^2 = p_n,$$

and thus $z$ conjugates $(e_0, \ldots, e_n)$ to $(p_0, \ldots, p_n)$. \hfill \square

**Proposition 2.25.** Every $n$-potent in a local $C^*$-algebra is similar to a normal $n$-potent.

**Proof.** Let $e$ be an $n$-potent in a local $C^*$-algebra. By Theorem ??, we can write

$$e = \sum_{k=1}^{n-1} \omega_k e_k,$$

where each $e_k$ is an idempotent. Proposition ?? gives us an invertible $z$ in $A$ such that $p_k = ze_kz^{-1}$ is a projection for all $k$. Then

$$zez^{-1} = \sum_{k=1}^{n-1} \omega_k p_k$$

is a normal $n$-potent (by Lemma ??) similar to $e$. \hfill \square

**Definition 2.26.** Let $e$ and $f$ be idempotents in a local $C^*$-algebra $A$. We say $e$ and $f$ are unitarily equivalent and write $e \sim_u f$ if there exists a unitary element $u$ in $A$ such that $ueu^* = f$. 

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**Proposition 2.27.** Let \( e \) and \( f \) be idempotents in a local \( C^* \)-algebra \( A \).

(a) If \( e \sim u f \), then \( e \sim s f \).

(b) If \( e \) and \( f \) are normal and \( e \sim s f \), then \( e \sim u f \).

**Proof.** Because \( u^* = u^{-1} \) for \( u \) unitary, (a) is immediate. To show (b), choose an invertible \( z \) in \( A \) so that \( zez^{-1} = f \), and use Theorem ?? and Proposition ?? to write

\[
e = \sum_{k=1}^{n-1} \omega_k p_k \quad \text{and} \quad f = \sum_{k=1}^{n-1} \omega_k q_k
\]

for projections \( p_0, p_1, \ldots, p_{n-1} \) and \( q_0, q_1, \ldots, q_{n-1} \). Then \( zp_k z^{-1} = q_k \) for \( 0 \leq k \leq n - 1 \).

Let \( z = u|z| \) be the polar decomposition of \( z \). Because \( zp_k = q_k z \) and (by taking adjoints) \( p_k z^* = z^* q_k \), we have

\[
|z|^2 q_k = zz^* q_k = zp_k z^* = q_k z z^* = q_k |z|^2.
\]

Similarly, \( |z|^2 p_k = p_k |z|^2 \). By the functional calculus,

\[
up_k u^* = up_k |z| |z|^{-1} u^* = u|z| p_k |z|^{-1} u^* = zp_k z^{-1} = q_k
\]

for all \( k \), and thus

\[
ueu^* = \sum_{k=1}^{n-1} \omega_k (up_k u^*) = \sum_{k=1}^{n-1} \omega_k q_k = f,
\]

as desired. \(\square\)

### 3. \( K_0 \)-theory with \( n \)-potents

We can now proceed to construct our \( n \)-potent \( K \)-theory groups.

**Definition 3.1.** Let \( R \) be a ring. For all \( k \geq 1 \), let \( \mathcal{P}^n_k(R) \) denote the set of \( n \)-potents in \( M_k(R) \), and let \( i_k \) denote the inclusion

\[
i_k(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}
\]

of \( M_k(R) \) into \( M_{k+1}(R) \), as well as its restriction as a map from \( \mathcal{P}^n_k(R) \) to \( \mathcal{P}^n_{k+1}(R) \). Define \( M_\infty(R) \) and \( \mathcal{P}^n_\infty(R) \) to be the (algebraic) direct limits

\[
M_\infty(R) = \bigcup_{k=1}^\infty M_k(R), \quad \mathcal{P}^n_\infty(R) = \bigcup_{k=1}^\infty \mathcal{P}^n_k(R) = \mathcal{P}^n(M_\infty(R)).
\]
We define a binary relation $\oplus$ on $\mathcal{P}_\infty^n(R)$ as follows: let $e$ and $f$ be elements of $\mathcal{P}_\infty^n(R)$, choose the smallest natural numbers $k$ and $\ell$ such that $e \in M_k(R)$ and $f \in M_\ell(R)$, and set
\[
e \oplus f = \text{diag}(e, f) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in \mathcal{P}_{k+1}^n(R) \subset \mathcal{P}_\infty^n(R).
\]

**Definition 3.2.** Let $R$ be a ring, and define an equivalence relation $\sim$ on $\mathcal{P}_\infty^n(R)$ as follows: take $e$ and $f$ in $\mathcal{P}_\infty^n(R)$, and choose a natural number $k$ sufficiently large that $e$ and $f$ are elements of $\mathcal{P}_k^n(R)$. Then $e \sim f$ if $e \sim_a f$ in $M_k(R)$. We let $\mathcal{V}_n(R)$ denote the set of equivalence classes of $\sim$.

Note that if $e = ab$ and $f = ba$ in $M_k(R)$, then
\[
\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}
\]
and
\[
\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},
\]
and therefore the equivalence relation described in Definition 3.2 is well-defined.

Note that for any $n$-potent $e, f$ in $M_\infty(R)$, we get
\[
e = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \sim \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = f.
\]

Thus, the binary operation $\oplus$ induces a binary operation $+$ on $\mathcal{V}_\infty^n(R)$ as follows: take $e$ and $f$ in $\mathcal{P}_\infty^n(R)$, and define
\[
[e] + [f] = [e \oplus f] = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.
\]
This operation is well-defined and commutative by Propositions 3.1 and 3.2.

The next proposition is straightforward and left to the reader.

**Proposition 3.3.** For every ring $R$ and natural number $n \geq 2$, $\mathcal{V}_n(R)$ is an abelian monoid under the addition defined above, and whose identity element is the class of the zero $n$-potent. If $\alpha : R \longrightarrow S$ is a unital ring homomorphism, then the induced map $\mathcal{V}_n(\alpha) : \mathcal{V}_n(R) \longrightarrow \mathcal{V}_n(S)$ given by
\[
\mathcal{V}_n(\alpha)([(a_{ij})]) = [(\alpha(a_{ij}))]
\]
is a well-defined homomorphism of abelian semigroups. The correspondence $R \mapsto \mathcal{V}_n(R)$ is a covariant functor from the category of rings and ring homomorphisms to the category of abelian monoids and monoid homomorphisms.
Definition 3.4. Let $R$ be a ring and let $n \geq 2$ be a natural number. We define $K_n^0(R)$ to be the Grothendieck completion $[?]$ of the abelian monoid $V_n(R)$. Given an $n$-potent $e$ in $P_n^\infty(R)$, we denote its class in $K^0_n(R)$ by $[e]$.

In light of Propositions ?? and ??, we could have alternately used similarity to define $V_n(R)$, and hence $K^0_n(R)$. If $A$ is a local Banach algebra, Proposition ?? and Corollary ?? imply that we obtain the same semigroup using homotopy as our equivalence relation, and if $A$ is a local $C^*$-algebra, unitary equivalence also gives us $V_n(A)$ by Propositions ?? and ??.

Proposition 3.5. The assignment $R \mapsto K^0_n(R)$ determines a covariant functor from the category of rings and ring homomorphisms to the category of abelian groups and group homomorphisms.

Proof. Proposition ?? states that $V$ is a covariant functor from the category of rings to the category of abelian monoids, and Grothendieck completion determines a covariant functor from the category of abelian monoids to the category of abelian groups; we get the desired result by composing these two functors. \hfill \Box

The following result shows that for (unital) algebras over a field of characteristic $\neq 2$, the tripotent $K$-theory functor $K^3_0$ offers us no new invariants over ordinary idempotent $K$-theory. However, we will see later (Theorem ??) that the situation is subtly different for $K^4_0$.

Theorem 3.6. Let $F$ be a field with characteristic $\neq 2$. If $A$ is a unital algebra over $F$ then there is a natural isomorphism

$$K^3_0(A) \cong (K_0(A))^2$$

of abelian groups.

Proof. If $e = e^3 \in M_\infty(A)$ is a tripotent, then one can easily check that

$$e_1 = \frac{1}{2}(e^2 + e) \quad \text{and} \quad e_2 = \frac{1}{2}(e^2 - e)$$

are (unique) idempotents in $M_\infty(A)$ such that $e = e_1 - e_2$. It follows that we have a natural bijection of abelian monoids

$$V^3(A) \to V^2(A) \oplus V^2(A)$$

$$[e] \mapsto [e_1] \oplus [e_2]$$

with inverse map $[e_1] \oplus [e_2] \mapsto [e_1 + e_2]$. Since these maps are additive, the result easily follows. \hfill \Box
While $K^n(\mathbb{P}(R))$ is well-defined for any ring $R$, to obtain a well-behaved theory where the usual exact sequences exist, we must restrict our attention to a smaller class of rings. The problem is that unlike the situation for idempotents, it is not generally true that if $e$ is an $n$-potent, then so is $1 - e$. However, given an $n$-potent in an algebra over the cyclotomic field $\mathbb{Q}(n - 1)$, there is an adequate substitute:

**Definition 3.7.** Let $e$ be an $n$-potent in a $\mathbb{Q}(n - 1)$-algebra $A$, and write

$$e = \sum_{k=1}^{n-1} \omega_k e_k$$

as in the conclusion of Theorem 2. We define an $n$-potent

$$e^\perp = \sum_{k=1}^{n-1} \text{diag}(\omega_1(1 - e_1), \omega_2(1 - e_2), \ldots, \omega_{n-1}(1 - e_{n-1})) \in M_{n-1}(A)$$

and call $e^\perp$ the complementary $n$-potent of $e$.

Observe that if $n = 2$, this definition agrees with the usual one for idempotents; i.e., $e^\perp = 1 - e$. Note also that $e \oplus e^\perp \sim_\lambda \omega$, where

$$\omega = \text{diag}(\omega_1 1_A, \ldots, \omega_n 1_A) \in M_{n-1}(\mathbb{Q}(n - 1)) \subseteq M_{n-1}(A).$$

**Proposition 3.8** (Standard picture of $K^n(\mathbb{P}(A))$). Let $n \geq 2$ be a natural number and let $A$ be a $\mathbb{Q}(n-1)$-algebra. Then every element of $K^n(\mathbb{P}(A))$ can be written in the form $[e] - [\omega]$, where $e$ in an $n$-potent in $M_0(A)$ for some natural number $k$ and $\omega$ is a diagonal $n$-potent in $M_k(\mathbb{Q}(n - 1))$.

**Proof.** Start with an element $[\tilde{e}] - [\tilde{f}]$ in $K^n(\mathbb{P}(A))$, and take $\tilde{f}^\perp$ to be the complementary $n$-potent of $f$ as defined in Definition 2. Then

$$[\tilde{e}] - [\tilde{f}] = ([\tilde{e}] + [\tilde{f}^\perp]) - ([\tilde{f}] + [\tilde{f}^\perp]).$$

The $n$-potents $\tilde{f}$ and $\tilde{f}^\perp$ are orthogonal, and therefore

$$[\tilde{f}] + [\tilde{f}^\perp] = [\tilde{f} + \tilde{f}^\perp] = [\omega],$$

where $\omega$ has the desired form. Finally we take $e$ to be $\tilde{e} \oplus \tilde{f}^\perp$, and by enlarging the matrix $\omega$, we obtain the desired result. □

**Proposition 3.9.** Let $n \geq 2$ and let $A$ be a $\mathbb{Q}(n-1)$-algebra. Suppose $e$ and $f$ are $n$-potents in $M_\infty(A)$. Then $[e] = [f]$ in $K^n(\mathbb{P}(A))$ if and only if $e \oplus \omega$ is similar to $f \oplus \omega$ for some $n$-potent $\omega$ in $M_\infty(\mathbb{Q}(n - 1))$.

**Proof.** The “only if” direction is obvious. To show the inference in the opposite direction, suppose that $[e] = [f]$ in $K^n(\mathbb{P}(A))$. By the definition of the Grothendieck completion, $e \oplus \tilde{e}$ is similar to $f \oplus \tilde{e}$ for some $n$-potent $\tilde{e}$ in $M_\infty(A)$. Then $e \oplus \tilde{e} \oplus \tilde{e}^\perp$ is similar to $f \oplus \tilde{e} \oplus \tilde{e}^\perp$. But if we
write \( \bar{e} = \sum_{k=1}^{n-1} \omega_k \tilde{e}_k \) as in Theorem 3.9, then Proposition 3.10(b) implies that

\[
\bar{e} \sim s \operatorname{diag}(\omega_1 \tilde{e}_1, \omega_2 \tilde{e}_2, \ldots, \omega_{n-1} \tilde{e}_{n-1}).
\]

Therefore \( \bar{e} \oplus \bar{e}^\perp \) is similar to an \( n \)-potent in \( M_\infty(Q(n-1)) \), and the proposition follows. \( \square \)

We next turn our attention to \( n \)-potent \( K \)-theory for nonunital algebras. Given a nonunital \( Q(n-1) \)-algebra \( A \), we define its unitization \( A^+ \) as the unital \( Q(n-1) \)-algebra \( A^+ = \{(a, \lambda) : a \in A, \lambda \in Q(n-1)\} \), where addition and scalar multiplication are defined componentwise, and multiplication is given by \((a, \lambda)(b, \tau) = (ab + a\tau + b\lambda, \lambda\tau)\).

**Definition 3.10.** Let \( A \) be a nonunital \( Q(n-1) \)-algebra, and let \( A^+ \) be its unitization. Let \( \pi : A^+ \to Q(n-1) \) be the algebra homomorphism \( \pi(a, \lambda) = \lambda \). Then we define \( K^0_0(A) = \ker \pi^* \).

It is easy to see that \( \pi^* \) is surjective, so by definition of \( K^0_0(A) \) we have a short exact sequence

\[
0 \to K^0_0(A) \to K^0_0(A^+) \xrightarrow{\pi^*} K^0_0(Q(n-1)) \to 0
\]

with splitting induced by the map \( \psi : Q(n-1) \to A^+ \) defined by \( \psi(\lambda) = (0, \lambda) \). In addition, it is easy to check that if \( A \) already has a unit and we form \( A^+ \), then \( \ker \pi^* \) is naturally isomorphic to our original definition of \( K^0_0(A) \).

**Proposition 3.11.** Let \( A \) be a nonunital \( Q(n-1) \)-algebra. Then every element of \( K^0_0(A) \) can be written in the form \([e] - [s(e)]\), where \( e \) is an \( n \)-potents in \( M_k(A^+) \) for some integer \( k \geq 1 \), and \( s = \psi \circ \pi : A^+ \to A^+ \) is the scalar mapping [3, Sect. 4.2.1].

**Proof.** Follows directly from Proposition 3.10 and Definition 3.10. \( \square \)

**Proposition 3.12** (Half-exactness). Every short exact sequence

\[
0 \to I \xrightarrow{i} A \xrightarrow{q} A/I \to 0
\]

of \( Q(n-1) \)-algebras, with \( A \) unital, induces an exact sequence

\[
K^0_0(I) \xrightarrow{i^*} K^0_0(A) \xrightarrow{q^*} K^0_0(A/I)
\]

of abelian \( n \)-potent \( K \)-theory groups.

**Proof.** Because \( q \circ i = 0 \), we have by functoriality that \( q_\ast \circ i_\ast = 0 \) and so the image of \( K^0_0(I) \) under \( i_\ast \) in \( K^0_0(A) \) is contained in the kernel of \( q_\ast \). To show the reverse inclusion, suppose we have \([e] - [\omega]\) in \( K^0_0(A) \)
such that \( q_*([e] - [\omega]) = 0 \). Then \([q(e)] = [q(\omega)] = [\omega]\) in \( K^n_0(A/I) \). By Proposition 3.4.2, there exists an \( n \)-potent \( \tau \) in \( M_\infty(\mathbb{Q}(n - 1)) \) so that
\[
q(e) \oplus \tau \sim_s \omega \oplus \tau.
\]
Choose \( N \) sufficiently large so that we may view \( e, \omega, \) and \( \tau \) as \( N \) matrices, and choose \( z \) in \( GL_{2N}(A/I) \) so that
\[
z(q(e) \oplus \tau)z^{-1} = \omega \oplus \tau.
\]
By Proposition 3.4.2 and Corollary 3.4.4 in [?], we can lift \( \text{diag}(z, z^{-1}) \) to an element \( u \) in \( GL_{4N}(A) \). Set \( f = u(e \oplus \tau)u^{-1} \). Then
\[
q(f) = \text{diag}(z, z^{-1})(q(e) \oplus \tau)\text{diag}(z^{-1}, z) = \omega \oplus \tau,
\]
and thus \( f \) and \( \omega \oplus \tau \) are in \( M_{4N}(I^+) \). Therefore
\[
[e] - [\omega] = [e \oplus \tau] - [\omega \oplus \tau] = i_*([f] - [\omega \oplus \tau])
\]
is in the image of \( K^n_0(I) \) under \( i_* \) as desired. \( \square \)

While it is not at all obvious from its definition, \( K^n_0(A) \) can be identified with a more familiar object.

**Theorem 3.13.** Let \( n \geq 2 \) be a natural number and let \( A \) be a not necessarily unital \( \mathbb{Q}(n - 1) \)-algebra. Then there is a natural isomorphism
\[
K^n_0(A) \cong (K_0(A))^{n-1}
\]
of abelian groups.

**Proof.** First consider the case where \( A \) is unital. We define a homomorphism \( \tilde{\psi} : \mathcal{V}^n(A) \rightarrow (\mathcal{V}_0(A))^{n-1} \) in the following way: for each \( n \)-potent \( e = \sum_{k=1}^{n-1} \omega_k e_k \) in \( M_\infty(A) \), set
\[
\tilde{\psi}[e] = ([e_1], [e_2], \ldots, [e_{n-1}]).
\]
It is easy to check that \( \tilde{\psi} \) is additive and well-defined. Next, define a homomorphism \( \tilde{\phi} : (\mathcal{V}(A))^{n-1} \rightarrow \mathcal{V}^n(A) \) by the formula
\[
\tilde{\phi}([f_1], [f_2], \ldots, [f_{n-1}]) =
\[
[\omega_1 \text{diag}(f_1, 0, 0, \ldots, 0) + \omega_2 \text{diag}(0, f_2, 0, \ldots, 0) + \cdots
+ \omega_{n-1} \text{diag}(0, 0, \ldots, 0, f_{n-1})].
\]
Note that
\[
\tilde{\psi}\tilde{\phi}([f_1], [f_2], \ldots, [f_{n-1}])
= \psi[\omega_1 \text{diag}(f_1, 0, 0, \ldots, 0) + \cdots + \omega_{n-1} \text{diag}(0, 0, \ldots, f_{n-1})]
\]
\[
= ([\text{diag}(f_1, 0, 0, \ldots, 0)], [\text{diag}(0, f_2, 0, \ldots, 0)] \ldots [\text{diag}(0, 0, \ldots, f_{n-1})])
\]
\[
= ([f_1], [f_2], \ldots, [f_{n-1}])
\]
and
\[ \tilde{\phi} \tilde{\psi}[e] = \phi([e_1], [e_2], \ldots, [e_{n-1}]) = [\omega_1 \text{diag}(e_1, 0, \ldots, 0) + \cdots + \omega_{n-1} \text{diag}(0, 0, \ldots, e_{n-1})] = [\text{diag}(\omega_1 e_1, \omega_2 e_2, \ldots, \omega_{n-1} e_{n-1})] = [e], \]
where the last equality is a consequence of Proposition ??(b). The universal mapping property of the Grothendieck completion implies that \( \tilde{\psi} \) extends uniquely to an abelian group isomorphism
\[ \psi : K_0^n(A) \longrightarrow (K_0(A))^n, \]
and thus the theorem is true for unital \( \mathbb{Q}(n-1) \)-algebras.

Now suppose that \( A \) does not have a unit. Then we have the following commutative diagram with exact rows:
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0^n(A) & \longrightarrow & K_0^n(A^+) & \longrightarrow & K_0^n(\mathbb{Q}(n-1)) & \longrightarrow & 0 \\
& & \text{\textasciitilde} & & \text{\textasciitilde} & & \text{\textasciitilde} & & \\
0 & \longrightarrow & K_0(A)^{n-1} & \longrightarrow & K_0(A^+)^{n-1} & \longrightarrow & K_0(\mathbb{Q}(n-1))^{n-1} & \longrightarrow & 0 \\
\end{array}
\]

An easy diagram chase shows that there is a unique group isomorphism from \( K_0^n(A) \) to \((K_0(A))^{n-1}\) that makes the diagram commute. \( \square \)

Since a complex algebra is a \( \mathbb{Q}(n-1) \)-algebra for all values of \( n \), we have the following immediate corollary.

**Corollary 3.14.** If \( A \) is a \( \mathbb{C} \)-algebra, there are natural isomorphisms
\[ K_0^n(A) \cong (K_0(A))^{n-1} \]
of abelian groups for all natural numbers \( n \geq 2 \).

We now arrive at the result that suggests why we should consider all \( K_0^n \)-functors for algebras over a cyclotomic field.

**Theorem 3.15.** Let \( \mathbb{Q}(4) = \mathbb{Q}[i] \) be the 4th cyclotomic field. Then we have the following isomorphisms of abelian groups:
\[
\begin{aligned}
K_0^2(\mathbb{Q}(4)) & \cong \mathbb{Z}, \\
K_0^3(\mathbb{Q}(4)) & \cong \mathbb{Z}^2, \\
K_0^4(\mathbb{Q}(4)) & \cong \mathbb{Z} \oplus 2\mathbb{Z}, \\
K_0^5(\mathbb{Q}(4)) & \cong \mathbb{Z}^4. \\
\end{aligned}
\]
Thus, \( K_0^4(\mathbb{Q}(4)) \not\cong \mathbb{Z}^3 \cong K_0^4(\mathbb{Q}(3)) \).
Proof. Since \( \mathbb{Q}(4) \) is a field \([?]\), we have \( K_0^2(\mathbb{Q}(4)) = K_0(\mathbb{Q}(4)) \cong \mathbb{Z} \). The field \( \mathbb{Q}(4) \) has characteristic \( 0 \neq 2 \), so Theorem \( ?? \) implies that \( K_0^2(\mathbb{Q}(4)) \cong (K_0(\mathbb{Q}(4))^2 \cong \mathbb{Z}^2 \). Theorem \( ?? \) implies that we have an isomorphism \( K_0^3(\mathbb{Q}(4)) \cong (K_0(\mathbb{Q}(4)))^4 \cong \mathbb{Z}^4 \).

However, the spectrum of 4-potents is contained in

\[
\Omega_3 = \left\{ 0, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} + \frac{\sqrt{3}}{2} i \right\} \subset \mathbb{C}
\]

which is not contained in \( \mathbb{Q}(4) \) since the two primitive 3rd roots of unity \( \omega = \zeta_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \) and \( \bar{\omega} = \bar{\zeta}_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \) are not in \( \mathbb{Q}(4) = \mathbb{Q}[i] \).

Given any 4-potent \( e \in M_n(\mathbb{Q}(4)) \subset M_n(\mathbb{C}) \) we can uniquely write

\[
e = e_1 + \omega e_2 + \bar{\omega} e_3,
\]

where \( e_1, e_2, e_3 \) are orthogonal idempotents in \( M_n(\mathbb{C}) \) that sum to an idempotent \( e_1 + e_2 + e_3 = e^3 \) in \( M_n(\mathbb{Q}(4)) \) by Lemma \( ?? \). We thus have that

\[
e^2 = e_1 + \omega e_2 + \omega e_3
\]

\[
e^3 = e_1 + e_2 + e_3
\]

because \( \omega^2 = \bar{\omega}, \omega^2 = \omega \), and \( \omega^3 = \bar{\omega}^3 = 1 \). Since \( \omega + \bar{\omega} = -1 \), this implies that the first idempotent

\[
e_1 = (e + e^2 + e^3)/3 \in M_n(\mathbb{Q}(4))
\]

and the sum of the last two idempotents

\[
e_2 + e_3 = e^3 - e_1 \in M_n(\mathbb{Q}(4))
\]

are both in \( M_n(\mathbb{Q}(4)) \). Using a simple trace argument and the fact that \( \omega, \bar{\omega} \notin \mathbb{Q}(4) \), we conclude that

\[
\text{rank}(e_2) = \text{trace}(e_2) = \text{trace}(e_3) = \text{rank}(e_3),
\]

and so \( \text{rank}(e_2 + e_3) = \text{trace}(e_2 + e_3) = 2 \text{trace}(e_2) \) is even. We then have a well-defined map

\[
\mathcal{V}^4(\mathbb{Q}(4)) \to \mathcal{V}^2(\mathbb{Q}(4)) \oplus 2\mathcal{V}^2(\mathbb{Q}(4)) \cong \mathbb{N} \oplus 2\mathbb{N}
\]

\[
[e] \mapsto [e_1] \oplus [e_2 + e_3] \cong \text{trace}(e_1) \oplus 2 \text{trace}(e_2);
\]

this is because the classes of \( e_1 \) and \( e_1 + e_2 \) are preserved by (stable) similarity, and the only invariant of an idempotent in a matrix ring over a number field (or a PID) is the rank (= trace). It is easy to check that this map is injective (using \( e_1 \perp e_2 + e_3 \) in \( M_n(\mathbb{Q}(4)) \)) and additive. The only question is surjectivity. It suffices to show that
there is a 4-potent $e$ over $\mathbb{Q}(4)$ whose stable similarity class is mapped to the generator $1 \oplus 2$ of $\mathbb{N} \oplus 2\mathbb{N}$. Consider the block diagonal matrix

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & -1 \end{pmatrix} \in M_3(\mathbb{Q}(4)),$$

which is easily checked to be quadripotent. The lower right quadripotent $2 \times 2$ invertible block has the desired eigenvalues $\omega$ and $\bar{\omega}$, and so does not diagonalize over $\mathbb{Q}(4)$. The result now follows easily. \qed

4. $n$-Homomorphisms and $K_0^n$ Functorality

We know from Proposition ?? that $K_0^n$ is a covariant functor from the category of (unital) rings and ring homomorphisms to the category of abelian groups and group homomorphisms. However, $K_0^n$ is actually functorial for a more general class of ring mappings.

**Definition 4.1.** Let $R$ and $S$ be rings. An additive map (not necessarily unital) $\phi : R \to S$ is called an $n$-homomorphism if

$$\phi(a_1a_2 \cdots a_n) = \phi(a_1)\phi(a_2) \cdots \phi(a_n)$$

for all $a_1, a_2, \ldots, a_n$ in $R$.

Obviously every (ring) homomorphism is an $n$-homomorphism, but the converse is false in general. For example, an $AE_n$-ring is a ring $R$ such that every additive map $\phi : R \to R$ is an $n$-homomorphism. Feigelstock [?, ?] classified all unital $AE_n$-rings. The algebraic version of $n$-homomorphism was introduced for complex algebras in [?] and has been carefully studied in the case of $C^*$-algebras in [?].

**Proposition 4.2.** Let $\phi : R \to S$ be an $n$-homomorphism between unital rings. Then $\phi$ induces a group homomorphism

$$\phi_* : K_0^n(R) \to K_0^n(S).$$

Furthermore, the assignment $R \mapsto K_0^n(R)$ is a covariant functor from the category of unital rings and $n$-homomorphisms to the category of abelian groups and ordinary group homomorphisms.

**Proof.** For each natural number $k$, we extend $\phi$ to a map from $M_k(R)$ to $M_k(S)$ by applying $\phi$ to each matrix entry; it is easy to check this also gives us an $n$-homomorphism. Moreover, $\phi$ is compatible with stabilization of matrices; the only nonobvious point to check is that $\phi$ respects algebraic equivalence.
Let $e$ and $f$ be algebraically equivalent $n$-potents in $M_k(R)$ for some $k$, and choose $a$ and $b$ in $M_k(R)$ so that $e = ab$ and $f = ba$. Define elements $a' = \phi(ea)\phi(f)^{n-2}$ and $b' = \phi(b)$ in $M_k(S)$. We compute:

$$a'b' = \phi(ea)\phi(f)^{n-2}\phi(b) = \phi((ea)f^{n-2}b) = \phi(ea(ba)^{n-2}b) = \phi(e(ab)^{n-1}) = \phi(e^n) = \phi(e).$$

A similar argument shows that $b'a' = \phi(f)$. Therefore $\phi$ determines a monoid homomorphism from $\mathcal{N}^n(R)$ to $\mathcal{N}^n(S)$, and hence a group homomorphism $\phi_* : K_0^a(R) \to K_0^a(S)$. We leave it to the reader to make the straightforward computations to show that we have a covariant functor.

Note that while we have an isomorphism $K_0^a(A) \cong (K_0(A))^{n-1}$ for $\mathbb{Q}(n-1)$-algebras, it is not at all clear from the right hand side of this isomorphism that $K_0^a(A)$ is functorial for $n$-homomorphisms.

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