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A Hopf index theorem for foliations ☆

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Abstract

We formulate and prove an analog of the Hopf Index Theorem for Riemannian foliations. We compute the basic Euler characteristic of a closed Riemannian manifold as a sum of indices of a non-degenerate basic vector field at critical leaf closures. The primary tool used to establish this result is an adaptation to foliations of the Witten deformation method.

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1. Introduction

The Euler characteristic is one of the simplest homotopy invariants of a smooth, closed manifold. We begin by briefly reviewing some standard theorems of topology which establish the equivalence of three ways of computing it. Then we proceed to new analogs for Riemannian foliations, where more details will be supplied.

For a smooth, closed manifold M, we define the Euler characteristic as

$$\chi(M) = \sum (-1)^k \dim H^k(M),$$

where H^k is de Rham cohomology. According to classical results of de Rham the spaces $H^k(M)$ are finite-dimensional and homotopy invariant; thus the definition of $\chi(M)$ makes sense and is homotopy invariant. An alternate route to the finite dimensionality of H^k and the topological invariance of χ goes

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by way of the Čech–de Rham double complex, which computes the de Rham cohomology and provides an isomorphism with the Čech cohomology of a finite good cover of M (here "good" means that nonempty finite intersections are diffeomorphic to $\mathbb{R}^n$). Using this isomorphism one can define the Euler class $e(E) \in H^n(M)$ of an oriented $(n-1)$-sphere bundle $E \to M$. The vanishing of the Euler class is a necessary, but not in general sufficient, condition for the bundle to have a section. In the case that e is the Euler class of the sphere bundle associated with the tangent bundle of M, $\int_M e = \chi(M)$. A corollary of this is the Hopf Index Theorem: $\chi(M)$ is the sum of the indices of the singular points of any nondegenerate vector field on M. Details can be found in [2].

Suppose that a smooth, closed manifold M is endowed with a smooth foliation $\mathcal{F}$. A form ω on M is *basic* if for every vector field X tangent to the leaves, $i(X)\omega = 0$ and $i(X)(d\omega) = 0$, where $i(X)$ denotes interior product with X. The exterior derivative of a basic form is again basic, so the basic forms are a subcomplex $\Omega_B^*(M, \mathcal{F})$ (or $\Omega_B^*(M)$) of the de Rham complex $\Omega^*(M)$. The cohomology of this subcomplex is the *basic cohomology* $H_B^*(M, \mathcal{F})$.

Suppose $\mathcal{F}$ has codimension q. The *basic Euler characteristic* is defined as

$$\chi_B(M, \mathcal{F}) = \sum_{k=0}^{q} (-1)^k \dim H_B^k(M, \mathcal{F}),$$

provided that all of the basic cohomology groups are finite-dimensional. Although $H_B^0(M, \mathcal{F})$ and $H_B^1(M, \mathcal{F})$ are always finite-dimensional, there are foliations for which higher basic cohomology groups can be infinite-dimensional. For example, in [6], the author gives an example of a flow on a 3-manifold for which $H_B^2(M, \mathcal{F})$ is infinite-dimensional. Therefore, we must restrict our investigation to a class of foliations for which the basic cohomology is finite-dimensional. A large class of foliations with this property are *Riemannian foliations*; a foliation is Riemannian if its normal bundle admits a holonomy-invariant Riemannian metric. There are various proofs that the basic cohomology of a Riemannian foliation on a closed manifold is finite-dimensional; see for example [4] for the original proof using spectral sequence techniques or [7] and [12] for proofs using a basic version of the Hodge theorem.

In Section 2 we develop a basic version of Čech–de Rham cohomology along the lines of [2]. The assumption that the foliation is Riemannian is needed to obtain basic partitions of unity and the basic Mayer–Vietoris sequence. For the basic Poincaré lemma we further assume that all of the leaves are closed. Examples are given to show that these conditions are necessary. The basic Čech–de Rham theorem establishes the equivalence of H_B^* and $\check{H}_B^*$, if all the leaves are closed. Examples show that the basic Čech cohomology and the basic de Rham cohomology are not necessarily isomorphic for Riemannian foliations in general.

A primary goal of this paper is to establish a foliation version of the Hopf Index Theorem. However, the standard proofs of this theorem do not carry over, even for Riemannian foliations. The problem is there are many Riemannian foliations that have a nonzero basic Euler characteristic, yet have trivial top-dimensional basic cohomology (for example, a non-taut Riemannian foliation [16] can possess this property). Thus it is impossible for these foliations to have any sort of basic Euler class that can be integrated to obtain the basic Euler characteristic (we do mention that for *taut* Riemannian foliations, one can define a nontrivial basic Euler class [18]).

To establish a Hopf Index Theorem for Riemannian foliations, another approach is required. The approach we use, which is in Section 3, involves a modification of the Witten deformation of the de Rham complex to the foliation case. Let V be a basic vector field on $(M, \mathcal{F})$; i.e., a vector field on M whose flow maps leaves to leaves (see [11] or [16], for example). In analogy with what is needed in the classical Hopf Index Theorem, we require that V satisfy a transverse nondegeneracy condition, which we call $\mathcal{F}$-nondegeneracy. We say that a leaf closure is *critical* for V if V is tangent to $\mathcal{F}$ at that leaf closure. When V is $\mathcal{F}$-nondegenerate, the critical leaf closures are necessarily isolated. For each critical leaf closure L, we define the *index* $\operatorname{ind}_L(V)$ of V at L; just as in the classical case, this index is always ± 1.

Next, let d_B be the restriction of the exterior derivative to basic forms, let δ_B denote the formal adjoint of d_B, and for each real number s, define the basic Witten differential

$$D_{B,s} = d_B + s i(V) + \delta_B + s V^\flat \wedge : \Omega_B^*(M) \to \Omega_B^*(M).$$

We show that the index of $D_{B,s}$ is independent of s, and examine the behavior of this operator as s goes to infinity. In the limit, the formula for the index of $D_{B,s}$ concentrates at the critical leaf closures. We next establish the necessary analytic properties of the basic Witten deformation. This leads to Theorem 3.18: Let $(M,\mathcal{F})$ be a Riemannian foliation, and let V be a basic vector field that is $\mathcal{F}$-nondegenerate. Given a critical leaf closure L, let $\mathcal{O}_L = \mathcal{O}_L(V)$ denote the orientation line bundle of V at L (Definition 3.1). Then

$$\chi_B(M,\mathcal{F}) = \sum_{L\text{ critical}} \operatorname{ind}_L(V)\chi_B(L,\mathcal{F},\mathcal{O}_L),$$

where $\chi_B(L,\mathcal{F},\mathcal{O}_L)$ is the alternating sum of the dimensions of the certain cohomology groups $H_B^*(L,\mathcal{F},\mathcal{O}_L)$ associated to the foliation $\mathcal{F}$ restricted to L (Definition 3.17). We remark that in many simple cases, $\chi_B(L,\mathcal{F},\mathcal{O}_L) = 1$, whence our formula takes on the precise form of the classical Hopf Index Theorem.

One important implication (Corollary 3.20) of this result is that if $(M,\mathcal{F})$ admits a basic vector field that is never tangent to the leaves, then $\chi_B(M,\mathcal{F}) = 0$. For example, this implies that if $\mathcal{F}$ has codimension 1 or 2 and has such a basic vector field, then M has infinite fundamental group (Corollary 3.21).

2. Basic Čech and de Rham cohomologies

Let M be a compact manifold, and let $\mathcal{F}$ be a Riemannian foliation on M. Let J be a finite ordered set, and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in J}$ be a finite basic open cover of M; i.e., a finite open cover in which each set U_α is a union of leaves. Such an open cover always exists in this case, because tubular neighborhoods of leaf closures are unions of leaves. For $\alpha_0,\alpha_1,\dots,\alpha_n$ an increasing sequence of indices define $U_{\alpha_0\alpha_1\dots\alpha_n} = \bigcap_{i=0}^n U_{\alpha_i}$, and let $\Omega_B^p(U_{\alpha_0\alpha_1\dots\alpha_n})$ be the collection of basic p-forms on $U_{\alpha_0\alpha_1\dots\alpha_n}$. Define

$$\delta : \prod \Omega_B^*(U_{\alpha_0\alpha_1\dots\alpha_k}) \to \prod \Omega_B^*(U_{\alpha_0\alpha_1\dots\alpha_{k+1}})$$

by the formula

$$(\delta\omega)_{\alpha_0\alpha_1\dots\alpha_{k+1}} = \sum_{i=0}^{k+1} (-1)^i \omega_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_{k+1}},$$

where each form is restricted to the appropriate subset. It is convenient to extend the index set of the product to include *all* sequences of the appropriate length from J, regardless of order or repetition, and to adopt the convention that interchanging indices of a component introduces a minus sign. So, for example, $\omega_{\alpha_2\alpha_1} = -\omega_{\alpha_1\alpha_2}$ for all α_1 and α_2 in J. It is straightforward to check that the definition of δ respects the sign convention and $\delta^2 = 0$.

Definition 2.1. A basic partition of unity subordinate to a basic open cover $\{U_\alpha\}$ is a partition of unity $\{\rho_\alpha\}$ consisting of basic functions ρ_α; that is, functions that are constant on leaves.

Lemma 2.2. *Every basic open cover of a Riemannian foliation admits a basic partition of unity.*

Proof. Endow the manifold with a bundle-like metric, and choose any partition of unity subordinate to the basic cover. Orthogonally project the functions in the partition of unity to the space of basic functions; the smoothness and other desired properties of the resulting functions are guaranteed by the results of [12]. □

Theorem 2.3 (Basic Mayer–Vietoris sequence). *The sequence*

$$0 \longrightarrow \Omega_B^*(M) \xrightarrow{r} \prod \Omega_B^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega_B^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \cdots$$

is exact, where r denotes the restriction map.

Proof. The proof of this theorem is the same as the proof of Theorem II.8.5 in [2], with forms replaced by basic forms and partitions of unity replaced by basic partitions of unity. □

Remark 2.4. If the foliation is not Riemannian, it may not admit basic partitions of unity, and in such cases, the Mayer–Vietoris sequence may fail to be exact. To describe such an example, we begin with some notation. For each real number t and open interval (a,b), let $L^t_{(a,b)}$ be the curve $\{(x, \frac{1}{x-a} + \frac{1}{b-x} + t) \mid a < x < b\}$ in $\mathbb{R}^2$. Foliate $[0,1) \times \mathbb{R}$ by the lines $x = 1/3$ and $x = 2/3$, along with Reeb components $L^t_{(0,1/3)}$, $L^t_{(1/3,2/3)}$, and $L^t_{(2/3,1)}$, where t ranges over all real numbers. Let $\mathbb{T}^2 \subset \mathbb{C}^2$ be the 2-torus, and consider the open map $p:[0,1) \times \mathbb{R} \to \mathbb{T}^2$ given by $p(x,y) = (\mathrm{e}^{2\pi i x}, \mathrm{e}^{2\pi i y})$. Then the image under p of our foliation on $[0,1) \times \mathbb{R}$ determines a foliation on $\mathbb{T}^2$. Let $U_0 = (0, 2/3) \times \mathbb{R}$, $U_1 = (1/3, 1) \times \mathbb{R}$, and $U_2 = ([0,1/3) \cup (2/3, 1)) \times \mathbb{R}$. There does not exist a basic partition of unity subordinate to the basic open cover $\{p(U_0), p(U_1), p(U_2)\}$ of $\mathbb{T}^2$, because the only basic functions are the constant ones. To see that the Mayer–Vietoris sequence is not exact in this example, first note that the intersection of all three sets in our basic open cover is empty. Therefore, for basic functions, we have the following piece of the Mayer–Vietoris sequence:

$$\begin{aligned} &C_B^\infty\big(p(U_0)\big) \oplus C_B^\infty\big(p(U_1)\big) \oplus C_B^\infty\big(p(U_2)\big) \\ &\quad \xrightarrow{\delta} C_B^\infty\big(p(U_0 \cap U_1)\big) \oplus C_B^\infty\big(p(U_1 \cap U_2)\big) \oplus C_B^\infty\big(p(U_0 \cap U_2)\big) \to 0. \end{aligned}$$

The map δ here is given by the formula $\delta(c_0, c_1, c_2) = (c_1 - c_0, c_2 - c_1, c_2 - c_0)$, which is not surjective since there is a nontrivial relation among the three components of the image of δ.

Remark 2.5. Even if a basic open cover of a (necessarily) non-Riemannian foliation does not admit basic partitions of unity, it is possible for the Mayer–Vietoris sequence to be exact. One such example is as follows; we maintain the notation of the previous remark. Foliate $[0,1) \times \mathbb{R}$ by leaves of two sorts: straight lines $L_a = \{(a, y) \mid y \in \mathbb{R}\}$ for $0 \leqslant a \leqslant 1/2$, and a Reeb component $L^t_{(1/2,1)}$, $t \in \mathbb{R}$. As in the previous remark, the image under p of this foliation of $[0,1) \times \mathbb{R}$ determines a foliation on $\mathbb{T}^2$. Let $U_0 = (0,1) \times \mathbb{R}$ and $U_1 = ([0,1/4) \cup (1/2,1)) \times \mathbb{R}$. Then there does not exist a basic partition of unity subordinate to the basic open cover $\{p(U_0), p(U_1)\}$ of $\mathbb{T}^2$, because every basic function on the Reeb component must be constant. In spite of this, it is straightforward to check that the Mayer–Vietoris sequence associated to this basic open cover is exact.

There are various cohomology theories that one can associate to the foliation $\mathcal{F}$. Note that the exterior derivative $d\omega$ of a basic form ω is basic, and therefore the collection of basic forms of $\mathcal{F}$ is a subcomplex of the de Rham complex. The cohomology of this subcomplex is denoted $H^*_B(M, \mathcal{F})$, and is called the *basic de Rham cohomology of* $(M, \mathcal{F})$.

Another cohomology theory can be defined for any basic open cover $\mathcal{U}$ of $(M, \mathcal{F})$. For each $p \geqslant 0$, define

$$C^p_B(\mathcal{U}, \mathbb{R}) = \ker\Big(d : \prod \Omega^0_B(U_{\alpha_0 \dots \alpha_p}) \to \prod \Omega^1_B(U_{\alpha_0 \dots \alpha_p})\Big).$$

It is easy to check that $(C^*_B(\mathcal{U}, \mathbb{R}), \delta)$ is a cochain complex; we declare its cohomology $\check{H}^p_B(\mathcal{U}, \mathbb{R})$ to be the *basic Čech cohomology* of the basic open cover $\mathcal{U}$.

Our immediate goal is to show that for "nice" covers $\mathcal{U}$, basic de Rham cohomology and basic Čech cohomology are isomorphic. To do this, we need a foliation version of the Poincaré lemma.

Proposition 2.6 (Basic Poincaré lemma). *Let $\mathcal{F}$ be a Riemannian foliation of M, suppose that all of the leaves of $\mathcal{F}$ are closed, and equip M with a bundle-like metric. Let L be a leaf; for $\varepsilon > 0$, let U be the tubular neighborhood of L consisting of points that are a distance less than ε from L. Then, for ε sufficiently small and for $k > 0$, every closed basic k-form on U is exact.*

Proof. First observe that since we have given M a bundle-like metric, the tubular neighborhood U is a union of leaves, and so the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to U makes sense. Choose ε small enough so that U misses the cut locus of L; since M is compact and L is closed, this can always be done. Fix $x \in L$, and let D be the exponential image of the ball of radius ε in the normal space $N_x L$. Then

$$\Omega^k_B(U) \cong \big\{\eta \in \Omega^k(D) \mid \eta \text{ is holonomy invariant}\big\}.$$

The holonomy of $\mathcal{F}$ acts by a finite subgroup Γ of the orthogonal group [11], so we have

$$\Omega^k_B(U) \cong \big\{\eta \in \Omega^k(D) \mid g^*\eta = \eta \text{ for all } g \text{ in } \Gamma\big\},$$

and this isomorphism commutes with the exterior derivative. Suppose $\omega \in \Omega^k_B(U)$ is closed, and let $\eta \in \Omega^k(D)$ be the closed form associated to ω via the isomorphism above. Since D is diffeomorphic to Euclidean space, there exists by the ordinary Poincaré lemma a form $\mu \in \Omega^{k-1}(D)$ such that $d\mu = \eta$. Now, μ may not be Γ-invariant, but the averaged form $\zeta = \frac{1}{|\Gamma|}\sum_{g \in \Gamma} g^*\mu$ is, and $d\zeta$ also equals η. Therefore η, and hence ω, is exact. □

Definition 2.7. A basic good cover of $(M, \mathcal{F})$ is a basic open cover $\mathcal{U}$ of $(M, \mathcal{F})$ with the feature that the basic cohomology of each intersection is trivial.

When all of the leaves of $\mathcal{F}$ are closed, the basic Poincaré lemma implies that we can obtain a basic good cover by covering each leaf by a sufficiently small tubular neighborhood. In fact, since M is compact, we can choose a *finite* basic good cover of M.

Theorem 2.8 (Basic Čech–de Rham theorem). *Suppose $\mathcal{F}$ is Riemannian and consists entirely of closed leaves, and suppose that $\mathcal{U}$ is a basic good cover of M. Then*

$$H_B^*(M,\mathcal{F}) \cong \check{H}_B^*(\mathcal{U},\mathbb{R}).$$

Proof. For each $p,q \geqslant 0$, let $C^p(\mathcal{U},\Omega_B^q)$ be the collection of basic q-forms restricted to $(p+1)$-fold intersections, and define a double complex

$$C^*(\mathcal{U},\Omega_B^*) = \bigoplus_{p,q\geqslant 0} C^p\big(\mathcal{U},\Omega_B^q\big).$$

The horizontal differential is the map δ defined above, and the vertical differential is $(-1)^p d$, where d is the exterior derivative. We compute the cohomology $H^*(\mathcal{U},\Omega_B^*)$ of this double complex in two ways.

First, augment $C^*(\mathcal{U},\Omega_B^*)$ by the column $\bigoplus_{q\geqslant 0}\Omega_B^q(M)$, and then map into $\bigoplus_{q\geqslant 0} C^1(\mathcal{U},\Omega_B^q)$ by the restriction map r. By Theorem 2.3, the rows of the augmented double complex are exact. By [2, p. 97], the cohomology $H^*(\mathcal{U},\Omega_B^*)$ of the original double complex is isomorphic to the cohomology of the initial column, which is precisely the basic de Rham cohomology $H_B^*(M,\mathcal{F})$ of $(M,\mathcal{F})$.

Second, augment the original double complex with the row $\bigoplus_{p\geqslant 0} C_B^p(\mathcal{U},\mathbb{R})$. Then the columns of the new double complex are exact, and thus $\check{H}_B^*(\mathcal{U},\mathbb{R})$ is also isomorphic to $H^*(\mathcal{U},\Omega_B^*)$. □

Corollary 2.9. *The group $\check{H}_B^p(\mathcal{U},\mathbb{R})$ is independent of the choice of basic good cover $\mathcal{U}$.*

Corollary 2.10. *If $\mathcal{F}$ is a Riemannian foliation and consists entirely of closed leaves, then $H_B^*(M,\mathcal{F})$ is finite-dimensional.*

In fact, the group $H_B^*(M,\mathcal{F})$ is finite-dimensional for any Riemannian foliation $\mathcal{F}$, as mentioned in the introduction.

Remark 2.11. The basic Čech–de Rham theorem is not necessarily true if the Riemannian foliation $\mathcal{F}$ has some non-closed leaves. Consider the foliation of the 2-torus $\mathbb{T}^2$ foliated by translates of a line of irrational slope. In this case, the only nonempty basic open set is $\mathbb{T}^2$ itself, so there is only one basic open cover $\mathcal{U}$ to choose. Clearly $\check{H}_B^1(\mathcal{U},\mathbb{R}) \cong 0$, whereas it is straightforward to check that $H_B^1(M,\mathcal{F}) \cong \mathbb{R}$. Note also that since $H_B^1(M,\mathcal{F})$ is nontrivial, the basic Poincaré lemma fails in this example.

3. Witten deformation of the basic de Rham complex

For the remainder of this paper, $\mathcal{F}$ is a Riemannian foliation on a smooth compact manifold M, and M is equipped with a bundle-like metric; ω_{vol} will denote the volume form associated to the metric.

Let V be a smooth vector field on $(M,\mathcal{F})$. We say that V is a *basic vector field* if for every vector field X in $T\mathcal{F}$, $[V,X]$ is in $T\mathcal{F}$. If in addition $V(x)$ is in $N_x\mathcal{F}$ (the normal space to the leaf at x) for every $x \in M$, we will say that V is a *basic normal vector field*. Basic normal vector fields always exist; the projection of any basic vector field onto $N\mathcal{F}$ is such a vector field. Associated to a basic vector field V is a one-parameter family of diffeomorphisms of M that preserves the foliation $\mathcal{F}$.

Let V be a basic vector field. Let L be a leaf closure with the property that V is tangent to every leaf in L; such a leaf closure will be called a *critical leaf closure* for V. We note in passing that if a basic vector field is tangent to the foliation at any point, it is in fact tangent to every leaf in the leaf closure. Define the *linear part* of V at $x \in L$ to be the linear map $V_L : N_x L \to N_x L$ defined by $X \mapsto \pi[V, \overline{X}]_x$, where $\overline{X}$ is any vector field that restricts to X at x, and $\pi : T_x M \to N_x L$ is the projection map. The basic vector field V will be called $\mathcal{F}$-*nondegenerate* if the linear part of V is an isomorphism at each point of every critical leaf closure. Every Riemannian foliation $\mathcal{F}$ admits $\mathcal{F}$-nondegenerate basic vector fields; simple examples are gradients of basic Morse functions (see [1]). A critical leaf closure for a $\mathcal{F}$-nondegenerate basic vector field V is necessarily isolated. We say that a nondegenerate vector field V has *index* 1 (respectively, *index* -1) at a critical leaf closure L if the determinant of the linear transformation V_L is everywhere positive (respectively, negative) on L. Clearly, if V is $\mathcal{F}$-nondegenerate, then at each critical leaf closure, V must have either index 1 or -1.

Given any point x_0 of a critical leaf closure L, choose orthonormal coordinates $\bar{y} = (y_1, \dots, y_{\bar{q}})$ for the normal space $N_x \overline{\mathcal{F}} = N_x L$, and extend these coordinates to orthonormal coordinates $y = (y_1, \dots, y_{\bar{q}}, y_{\bar{q}+1}, \dots, y_q)$ for $N_x \mathcal{F}$. Let $x = (x_1, \dots, x_p)$ be geodesic normal coordinates for the leaf near x_0. The coordinates (x, y) parametrize a tubular neighborhood of L near x_0 via the normal exponential map. It is elementary to check that we may write V near x_0 as an orthogonal sum $V = V_1 + V_2 + V_3$, where V_1 is tangent to $\mathcal{F}$, $V_2(y) = \sum_{i=1}^{\bar{q}} \alpha^i(y) \frac{\partial}{\partial y_i}$, and $V_3(y) = \sum_{j=\bar{q}+1}^{q} \beta^j(y) \frac{\partial}{\partial y_j}$. The linear transformation V_L is given by multiplication by the $\bar{q} \times \bar{q}$ matrix $(\frac{\partial \alpha^i}{\partial y_m}(y))_{1 \leqslant i,m \leqslant \bar{q}}$ on $N_x L$, so that the index of V is simply the sign of the determinant of that matrix. We remark that if V is the gradient of a basic function f, then V_L is the Hessian of f restricted to the normal ball, and $V_1 = V_3 = 0$.

Observe that a basic normal vector field V (hence $V_1 = 0$) is determined on a tubular neighborhood U of the leaf closure L by its values on a transverse $\bar{q}$-dimensional ball. Let $B(\delta)$ be the image of a ball of radius δ under the normal exponential map $\exp_x^{\perp} : N_x L \to U$ at $x \in L$, so that $y = (y_1, \dots, y_{\bar{q}})$ are exponential coordinates for this ball. The holonomy group at x is represented on $B(\delta)$ by a group Γ of orthogonal transformations (see [11]), so that we may identify basic normal vector fields on U with normal vector fields restricted to $B(\delta)$ that are equivariant with respect to the action of Γ. That is, for any $g \in \Gamma$, every basic normal vector field restricted to $B(\delta)$ has the form

$$V(y) = \sum_{i=1}^{q} v^i(\bar{y}) \frac{\partial}{\partial y_i} = \begin{pmatrix} v^1(\bar{y}) \\ \vdots \\ v^q(\bar{y}) \end{pmatrix}$$

and satisfies

$$g \begin{pmatrix} v^1(\bar{y}) \\ \vdots \\ v^q(\bar{y}) \end{pmatrix} = \begin{pmatrix} v^1(g\bar{y}) \\ \vdots \\ v^q(g\bar{y}) \end{pmatrix}.$$

Note that the notation $g\bar{y}$ denotes the restriction of the action of g on $N_x \mathcal{F}$ to the subspace $N_x L$. This makes sense, because elements of the holonomy group map vector fields orthogonal to L to other vector fields orthogonal to L. As a consequence, the action of Γ commutes with the action of V_L on $N_x L$.

Definition 3.1. Let V be a basic vector field that has a nondegenerate zero on a critical leaf closure L. At each point $x \in L$, let $V_{L,x} : N_x L \to N_x L$ be the linear part of V at x. Since $V_{L,x}$ is invertible, it can be written as the (uniquely determined) polar decomposition $V_{L,x} = P_x \Theta_x$, where $P_x = \sqrt{V_{L,x}^* V_{L,x}}$ is positive and symmetric, and $\Theta_x = P_x^{-1} V_{L,x}$ is an isometry. The orientation line bundle $\mathcal{O}_L = \mathcal{O}_L(V)$ of V at L is the orientation bundle of the subbundle of NL spanned by the eigenvectors of Θ_x corresponding the eigenvalue -1.

Remark 3.2. It can be shown that $\mathcal{O}_L$ is a smooth line bundle. Observe that in the context of this paper, the singularity of the subbundle is not an issue, since the eigenvalues of Θ_x are constant as x moves along the leaves. We also observe that if $V = \nabla f$ for a Bott–Morse function f, then this orientation line bundle corresponds to the orientation line bundle of the subbundle of negative directions of f.

We now prove a result that puts a basic vector field into standard form.

Lemma 3.3. *Let V be a basic vector field that is $\mathcal{F}$-nondegenerate, and let L be a critical leaf closure. Let $T_\delta(L)$ denote the tubular neighborhood of radius δ around L, and assume that δ is chosen so that $T_\delta(L)$ does not contain any other critical leaf closures and misses the cut locus of L. Then there is a basic normal vector field $\widetilde{V}$ and a $\tilde{\delta}$ with $0 < \tilde{\delta} < \delta$ such that*:

(1) *$\widetilde{V} = V$ outside $T_\delta(L)$;*
(2) *L is the only critical leaf closure of $\widetilde{V}$ in $T_\delta(L)$;*
(3) *The index of $\widetilde{V}$ at L is the same as the index of V at L;*
(4) *$\widetilde{V} = \nabla f$ on $T_{\tilde{\delta}}(L)$, where f is a basic function whose Morse index* (*restricted to a normal ball at a point of L*) *is even if the index of V is 1 at L and is odd if the index of V is -1 at L;*
(5) *The orientation line bundle of $\widetilde{V}$ at L is identical to the orientation line bundle of V at L.*

Proof. Choose coordinates $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_{\bar{q}}, y_{\bar{q}+1}, \ldots, y_q)$ near a point $x \in L$ as described in the paragraphs above. As before, we write V in the form $V = V_1 + V_2 + V_3$, where V_1 is tangent to $\mathcal{F}$, $V_2(y) = \sum_{i=1}^{\bar{q}} \alpha^i(y) \frac{\partial}{\partial y_i}$, and $V_3(y) = \sum_{j=\bar{q}+1}^{q} \beta^j(y) \frac{\partial}{\partial y_j}$. Given any δ' such that $0 < \delta' < \delta$, we may multiply the tangent component V_1 by a radial basic function that is zero in $T_{\delta'}(L)$ and is 1 outside $T_\delta(L)$. In doing so, we preserve the index and orientation line bundle and yet restrict to the case where V is a basic normal vector field. We now assume that V has already been modified in this way. Next, the basic normal vector field V is determined on $T_{\delta'}(L)$ by its restriction to a normal ball $B(\delta')$ with the $\bar{y} = (y_1, \ldots, y_{\bar{q}})$ coordinates, and we may write $V(\bar{y}) = \sum_{i=1}^{\bar{q}} v^i(\bar{y}) \frac{\partial}{\partial y_i} + \sum_{j=\bar{q}+1}^{q} v^j(\bar{y}) \frac{\partial}{\partial y_j}$ on $B(\delta')$. Again, we may multiply the component $\sum_{j=\bar{q}+1}^{q} v^j(\bar{y}) \frac{\partial}{\partial y_j}$ by a similar radial basic function if necessary so that this component vanishes on a given $B(\delta'')$ such that $0 < \delta'' < \delta'$, without changing the relevant properties of V. For the remainder of this proof, assume that we have already modified V so that it is in the form $V = \sum_{i=1}^{\bar{q}} v^i(\bar{y}) \frac{\partial}{\partial y_i}$ restricted to $B(\delta)$, choosing a slightly smaller δ if necessary. It follows that $V(\bar{y}) = V_L(\bar{y}) + \mathrm{O}(\|\bar{y}\|^2)$ on $B(\delta)$. Given $x \in L$, we write $V_L : N_x L \to N_x L$ in terms of its polar decomposition $V_L = P\Theta$. Let Γ be the closed subgroup of isometries on $N_x L$ induced from the representation of the holonomy group at x on $N_x L$. For every $g \in \Gamma$, $g V_L = V_L g$, whence $V_L^* g^{-1} = g^{-1} V_L^*$. This in turn implies that every $g \in \Gamma$ commutes with P and Θ. Let P_t be defined by $P_t v_i = \lambda_i(t) v_i$ on the λ_i-eigenspace of P, where each $\lambda_i(t)$ is any smooth positive function such that $\lambda_i(0) = \lambda_i$ and $\lambda_i(1) = 1$.

Since every $g \in \Gamma$ and P have simultaneous eigenspaces, every $g \in \Gamma$ also commutes with each P_t. Thus, the smooth, one-parameter family of transformations $\{T_t = P_t \Theta\}$ is a deformation of V_L $(t = 0)$ to an orthogonal transformation Θ $(t = 1)$ that has constant index (that is, the sign of the determinant of the linear transformation does not change). Next, since Θ is orthogonal, there is a complex orthogonal basis $\{w_k\}$ of $N_x L \otimes \mathbb{C}$ consisting of eigenvectors such that $\Theta w_k = \mathrm{e}^{i\theta_k} w_k$, where $0 \leqslant \theta_k < 2\pi$. If θ_k is 0, then Θ acts by the identity on $\operatorname{span}\{\operatorname{Re} w_k, \operatorname{Im} w_k\}$. If θ_k is π, then Θ multiplies each vector in $\operatorname{span}\{\operatorname{Re} w_k, \operatorname{Im} w_k\}$ by -1. If $\theta_k \neq 0$ and $\theta_k \neq \pi$, then Θ acts by a rotation of θ_k on $\operatorname{span}\{\operatorname{Re} w_k, \operatorname{Im} w_k\}$, which in this case is necessarily 2-dimensional. We let $\Theta_0 = \Theta$ and define the transformation Θ_t for $0 < t \leqslant 1$ by

$$\Theta_t(w_k) = \begin{cases} w_k & \text{if } \theta_k = 0, \\ -w_k & \text{if } \theta_k = \pi, \\ \mathrm{e}^{i(1-t)\theta_k} w_k & \text{otherwise.} \end{cases}$$

The smooth, one-parameter family of transformations $\{\Theta_t\}$ is a deformation of the orthogonal transformation Θ $(t = 0)$ to a transformation Θ_1 $(t = 1)$ that has constant index. Observe that since each $g \in \Gamma$ commutes with Θ_0, the obvious action of g on $N_x L \otimes \mathbb{C}$ satisfies $g(w_k) = \mathrm{e}^{i\alpha_k} w_k$ for some α_k; it follows that g commutes with each Θ_t. The final transformation Θ_1 may be described in a real orthogonal basis as a diagonal matrix whose diagonal consists of 1's and (-1)'s; this transformation is the linear part of a vector field of the form $\sum_{i=1}^{\bar{q}} \pm y^i \frac{\partial}{\partial y_i}$, where the y_i are geodesic normal coordinates on $\exp^{\perp}(N_x L)$ corresponding to that particular basis. We also observe that $f(\bar{y}) := \frac{1}{2} \sum_{i=1}^{\bar{q}} \pm (y^i)^2$ is then a basic function that is well-defined on a small neighborhood of L, and the linear part of $\nabla f(\bar{y})$ at $x \in L$ is $\Theta_1(\bar{y}) + \mathrm{O}(\|\bar{y}\|^2)$. Note that we may extend f to be a basic function on all of $(M, \mathcal{F})$ by multiplying by a radial cutoff function and extending by zero. Combining the two deformations described above, we see that V_L may be smoothly deformed to a Γ-equivariant transformation of the form Θ_1 in such a way that the index is unchanged throughout the deformation.

The argument that follows is somewhat similar to that found in [5]. Let $\{Y_t : N_x L \to N_x L\}$ be a smooth, one-parameter family of Γ-equivariant transformations constructed above such that $Y_0 = V_L$ and Y_1 is the linear part of $\sum_{i=1}^{\bar{q}} \pm y^i \frac{\partial}{\partial y_i}$. Thus each $Y_t(\bar{y})$ is a vector field that is well-defined in a sufficiently small tubular neighborhood of the leaf closure L. Next, let $Z_t(\bar{y}) = h(t)(\nabla f(\bar{y}) - Y_1(\bar{y}))$, with Y_1 and ∇f as above and $h : \mathbb{R} \to \mathbb{R}$ a smooth positive function such that $h(t) = 0$ for $t \leqslant 0$ and $h(t) = 1$ for $t \geqslant 1$. Because of the remarks above, $Z_t(\bar{y}) = \mathrm{O}(\|\bar{y}\|^2)$ for all t. Let η^1 be a radial (and therefore basic) cutoff function that is 1 in an r_1-neighborhood of L and 0 outside an $2r_1$-neighborhood of L (r_1 will be chosen shortly). Then for r_1 sufficiently small, $\eta^1(\bar{y})(Y_t(\bar{y}) + Z_t(\bar{y}))$ is a well-defined basic vector field on $(M, \mathcal{F})$. Let the basic vector field X_t^1 be defined by

$$X_t^1 = \eta^1(Y_t + Z_t) + \left(1 - \eta^1\right)V,$$

so that

$$X_t^1(\bar{y}) = \eta^1(\bar{y})(Y_t - Y_0)(\bar{y}) + Y_0(y) + \mathrm{O}\left(\|\bar{y}\|^2\right).$$

Observe that X_t^1 is a smooth, one-parameter family of basic vector fields that agree with V outside a $2r_1$-neighborhood of L, and $X_1^1 = \nabla f$ inside a r_1-neighborhood of L. Let $m_1 = \inf_{\|\bar{y}\|=1, 0 \leqslant t \leqslant 1} \|Y_t(\bar{y})\|$, and let $m_2 = \sup_{\|\bar{y}\|=1, 0 \leqslant t \leqslant 1} \|\frac{d}{dt} Y_t(\bar{y})\|$. Observe that $0 < m_1 < \infty$ and $0 \leqslant m_2 < \infty$. Therefore, if

$t_1 < \frac{m_1}{m_2} \leqslant \infty$, then

$$\begin{aligned}\left\|\eta^1(\bar y)(Y_t - Y_0)(\bar y) + Y_0(\bar y)\right\| &\geqslant \left\|Y_0(\bar y)\right\| - \left\|(Y_t - Y_0)(\bar y)\right\| \\ &\geqslant \left\|Y_0(\bar y)\right\| - t m_2 \\ &> 0\end{aligned}$$

for $t \leqslant t_1$, and hence $\eta^1(\bar y)(Y_t - Y_0)(\bar y) + Y_0(\bar y)$ is invertible. Choose $r_1 > 0$ so small that $\mathrm{O}(\|\bar y\|^2) < \eta^1(\bar y)(Y_t - Y_0)(\bar y) + Y_0(\bar y)$ for $\|\bar y\| < 2r_1$; then the leaf closure L is the only critical leaf closure of the basic vector field $X_t^1(\bar y)$ in the $2r_1$-neighborhood of L. We continue by defining for $t_k \leqslant t \leqslant t_{k+1} := t_k + t_1$

$$X_t^k = \eta^k(Y_t + Z_t) + \left(1 - \eta^k\right) X_t^{k-1} = \eta^k(\bar y)(Y_t - Y_{t_k})(\bar y) + Y_{t_k}(\bar y) + \mathrm{O}\left(\|\bar y\|^2\right),$$

where η^k is 1 in an r_k-neighborhood of L and 0 outside a $2r_k$-neighborhood of L (where $0 < r_k \leqslant r_{k-1}$ is chosen as above). Eventually, we will find a k such that $X_t^k = V$ outside a $2r_1$-neighborhood of L and is $X_t^k = Y_t + Z_t$ inside a r_k-neighborhood of L. Furthermore, L is the only critical leaf closure of X_t^k in a $2r_1$-neighborhood of L, and the index of X_t^k at L is the same as the index of V at L for $0 \leqslant t \leqslant 1$. Finally, $X_1^k = \nabla f$ inside a r_k-neighborhood of L.

Observe that if we let $2r_1 \leqslant \delta$, $\tilde\delta = r_k$, and $\widetilde V = X_1^k$, then the first four properties are satisfied. Moreover, since the orientation line bundles of Y_t and X_t^k at L are constant in t, the last property is satisfied as well. $\square$

We now proceed with a modified version of Witten's deformation of the de Rham complex (see [17] and [14]). Let V be a basic normal vector field, and let $i(V): \Omega_B^*(M) \to \Omega_B^*(M)$ denote interior multiplication with V. For a given $s > 0$, let $d_{B,s} = d_B + s i(V): \Omega_B^*(M) \to \Omega_B^*(M)$. Note that $d_{B,s}$ is the restriction of the differential operator $d_s = d + s i(V): \Omega^*(M) \to \Omega^*(M)$. The formal adjoint of $d_{B,s}$ is $\delta_{B,s} = \delta_B + s V^\flat\wedge: \Omega_B^*(M) \to \Omega_B^*(M)$, where δ_B is the basic adjoint of d_B and $V^\flat$ is the basic one-form $\langle V, \cdot\rangle$. From [12], we know that δ_B is the restriction of the differential operator $\delta + \varepsilon: \Omega^*(M) \to \Omega^*(M)$ to basic forms, where $\varepsilon: \Omega^i(M) \to \Omega^{i-1}(M)$ is a zeroth-order operator that involves mean curvature and Rummler's formula. The operator ε has the additional property that $P\varepsilon P = 0$, where $P: L^2(\Omega^*(M)) \to L^2(\Omega_B^*(M))$ is the orthogonal projection. We define

$$D_{B,s} = d_{B,s} + \delta_{B,s}: \Omega_B^*(M) \to \Omega_B^*(M).$$

Letting $D_B = d_B + \delta_B$ and $H = i(V) + V^\flat\wedge$, observe that

$$D_{B,s}^2 = (D_B + sH)^2 = D_B^2 + s(HD_B + D_BH) + s^2H^2.$$

The operator $H^2 = (i(V) + V^\flat\wedge)^2$ acts by multiplication by the basic function $\|V\|^2$. A simple calculation shows $Z' := H(d + \delta + \varepsilon) + (d + \delta + \varepsilon)H$ satisfies

$$\begin{aligned} Z' &= \mathcal{L}_V + (\mathcal{L}_V)^{*,B} + \left(d\left(V^\flat\right)\right)\wedge + \left(\left(d\left(V^\flat\right)\right)\wedge\right)^{*,B} \\ &= \mathcal{L}_V + (\mathcal{L}_V)^* + Z + \left(d\left(V^\flat\right)\right)\wedge + \left(\left(d\left(V^\flat\right)\right)\wedge\right)^*, \end{aligned} \tag{3.1}$$

where $\mathcal{L}_V = i(V)\,d + d\,i(V)$ denotes the Lie derivative in the V direction, the superscript $*, B$ denotes the adjoint restricted to basic forms, and $Z := \varepsilon \circ V^\flat\wedge + V^\flat\wedge \circ \varepsilon$ is a zeroth order operator. One can show using the Leibniz rule that $\mathcal{L}_V + (\mathcal{L}_V)^*$ commutes with multiplication by a function, so this operator is also an operator of order zero. Thus Z' is a differential operator of order zero, and it agrees with

$HD_B + D_BH$ on basic forms. Also note that Z' maps odd forms to odd forms and even forms to even forms.

Observe that $D_B^2 = \Delta_B^j = d_B\delta_B + \delta_B d_B$ on basic j-forms, the basic Laplacian. By the results of [12], this operator is essentially self-adjoint and has eigenvalues $0 \leqslant \lambda_1^{B,j} \leqslant \lambda_2^{B,j} \leqslant \lambda_3^{B,j} \leqslant \cdots$ with the property that $\lambda_k^{B,j} \geqslant Ck^{2/n}$ for some positive constant C and sufficiently large k (see [13] for more precise asymptotics). Furthermore, the basic Hodge decomposition theorem (see [7,12]) states that $\ker \Delta_B$ is finite-dimensional and that the space of basic j-forms decomposes orthogonally as $\operatorname{im} d_B \oplus \operatorname{im} \delta_B \oplus \ker \Delta_B^j$. Letting Δ_B^j denote the Laplacian on j-forms, we have $\ker \Delta_B^j \cong H_B^j(M, \mathcal{F})$. This implies

$$\begin{aligned}
\chi_B(M, \mathcal{F}) &= \sum_{j=0}^{q} (-1)^j \dim H_B^j(M, \mathcal{F}) \\
&= \sum_{j=0}^{q} (-1)^j \dim \ker \Delta_B^j \\
&= \operatorname{index}\bigl(D_B : \Omega_B^{\text{even}}(M) \to \Omega_B^{\text{odd}}(M)\bigr).
\end{aligned}$$

Next, let $K_B^j(t, x, y)$ denote the basic heat kernel on j-forms (see [7,12,13]), which is the fundamental solution of the basic heat equation. More specifically, consider the bundle $E \to [0, \infty) \times M \times M$ where $E_{(t,x,y)} = \operatorname{Hom}(\bigwedge^j (N_y\mathcal{F})^*, \bigwedge^j (N_x\mathcal{F})^*)$. The basic heat kernel is a section of E that is basic in x and y and satisfies

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \Delta_{B,x}^j\right) K_B^j(t, x, y) = 0 \quad \text{for } t > 0; \\
&\lim_{t \to 0^+} \int_{M_y} K_B^j(t, x, y)\beta(y)\omega_{\text{vol}}(y) = \beta(x) \quad \text{for all basic } j\text{-forms } \beta.
\end{aligned}$$

In [12, Theorem 3.5], the authors show that $K_B^j(t, x, y)$ exists, is smooth in x, y, and t, is unique, and satisfies

$$K_B^j(t, x, y) = P_x P_y \widetilde{K}^j(t, x, y) = \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_k^{B,j} t} \alpha_k(x) \otimes \alpha_k^*(y),$$

where the set of j-forms $\{\alpha_1, \alpha_2, \ldots\}$ is an orthonormal basis of basic eigenforms corresponding to the eigenvalues $\{\lambda_1^{B,j}, \lambda_2^{B,j}, \ldots\}$, and $\widetilde{K}^j(t, x, y)$ is the heat kernel corresponding to the strongly elliptic operator $\Delta^j + \delta\varepsilon^* + \varepsilon^*\delta$. The operator Δ^j is the ordinary Laplacian on j-forms. Note that this sum is finite if the leaves are dense. The map P_x is the orthogonal projection from $L^2(\Omega^*(M))$ to $L^2(\Omega_B^*(M))$ in the x-variable. The map P_y is the induced basic projection on duals of forms in the y-variable. In [12], the authors show that this basic projection P (or P_x or P_y) maps smooth forms to smooth basic forms, and the results in [12] also imply that the map $\beta \mapsto P\beta$ is smooth. The main other fact used in the proofs of the results concerning the basic Laplacian and the basic heat kernel is that Δ_B^j is the restriction of $\Delta^j + \varepsilon d + d\varepsilon$, which is a strongly elliptic operator defined on all forms.

The standard heat kernel approach to index calculations carries over to the basic case. Since D_B maps an eigenspace of Δ_B in $\Omega_B^{\text{even}}(M)$ isomorphically onto an eigenspace in $\Omega_B^{\text{odd}}(M)$, we have

$$\begin{aligned}\chi_B(M,\mathcal{F}) &= \operatorname{index}\bigl(D_B : \Omega_B^{\text{even}}(M) \to \Omega_B^{\text{odd}}(M)\bigr)\\ &= \sum_{j=0}^{q}(-1)^j \operatorname{tr}\bigl(\mathrm{e}^{-t\Delta_B^j}\bigr)\\ &= \sum_{j=0}^{q}(-1)^j \int_M \operatorname{tr} K_B^j(t,x,x)\omega_{\text{vol}}(x)\\ &= \int_M \operatorname{tr} K_B^{\text{even}}(t,x,x)\omega_{\text{vol}}(x) - \int_M \operatorname{tr} K_B^{\text{odd}}(t,x,x)\omega_{\text{vol}}(x).\end{aligned}$$

A similar analysis may be applied to $D_{B,s} = D_B + sH$. The operator $D_{B,s}^2 = D_B^2 + s(HD_B + D_B H) + s^2\|V\|^2$ is the restriction of the strongly elliptic operator

$$\Delta_s' = \Delta + \varepsilon d + d\varepsilon + sZ' + s^2\|V\|^2 \tag{3.2}$$

to basic forms, and the terms involving s are zeroth order. By a proof similar to that in [12], this operator is essentially self-adjoint and has eigenvalues that grow at a rate similar to those of Δ_B. The basic heat kernel $K_{B,s}^{\text{even}}(t,x,y)$ corresponding to $D_{B,s}^2$ exists, is smooth in x, y, and t, is unique, and satisfies

$$K_B^{\text{even}}(t,x,y) = P_x P_y \widetilde{K}_s^{\text{even}}(t,x,y) = \sum_{k=1}^{\infty} \mathrm{e}^{-t\lambda_k^{B,s,\text{even}}} \alpha_k^s(x) \otimes \bigl(\alpha_k^s(y)\bigr)^*,$$

where the set $\{\alpha_1^s, \alpha_2^s, \ldots\}$ is an orthonormal basis of even basic eigenforms corresponding to the eigenvalues $\{\lambda_1^{B,s,\text{even}}, \lambda_2^{B,s,\text{even}}, \ldots\}$ of $D_{B,s}^2$, and $\widetilde{K}_s^{\text{even}}(t,x,y)$ is the heat kernel corresponding to the strongly elliptic operator $\Delta^{\text{even}} + \delta\varepsilon^* + \varepsilon^*\delta + s(Z')^* + s^2\|V\|^2$. Similar results are true for $D_{B,s}^2$ on odd forms. Again, we have that $D_{B,s}$ maps an eigenspace of $D_{B,s}^2$ in $\Omega_B^{\text{even}}(M)$ isomorphically onto an eigenspace in $\Omega_B^{\text{odd}}(M)$, so that

$$\begin{aligned}&\operatorname{index}\bigl(D_{B,s} : \Omega_B^{\text{even}}(M) \to \Omega_B^{\text{odd}}(M)\bigr)\\ &\quad= \int_M \operatorname{tr} K_{B,s}^{\text{even}}(t,x,x)\omega_{\text{vol}}(x) - \int_M \operatorname{tr} K_{B,s}^{\text{odd}}(t,x,x)\omega_{\text{vol}}(x).\end{aligned} \tag{3.3}$$

The above results about $D_{B,s}^2$ imply results about the first order operator $D_{B,s}$. The spectrum of $D_{B,s}^2$ is nonnegative; let E_{λ^2} be an eigenspace of $D_{B,s}^2$ with eigenvalue λ^2. Then $(D_{B,s}+\lambda)E_{\lambda^2}$ and $(D_{B,s}-\lambda)E_{\lambda^2}$ are finite-dimensional orthogonal eigenspaces of $D_{B,s}$ with eigenvalues λ and $-\lambda$, respectively. Since $D_{B,s}$ is the restriction of the first order elliptic operator

$$D_s = d + \delta + \varepsilon + s\bigl(i(V) + V^\flat\wedge\bigr),$$

the subspaces $(D_{B,s} \pm \lambda)E_{\lambda^2}$ are spanned by smooth eigenforms. Moreover, there is an orthonormal basis of $L^2(\Omega_B^*(M))$ consisting of such smooth eigenforms.

The operators $\frac{\partial}{\partial t} - iD_s^*$ and $\frac{\partial}{\partial t} + iD_s^*$ are strongly hyperbolic, so our initial value problem has a unique solution (see [3, Sections 69–74], [8, Chapters IV–V], [10, Chapter 6], [15, Section 6.5]). Thus, we define the function $e^{itD_s^*}u$ to be the unique solution to the initial value problem (the *generalized traveling wave equation*)

$$\left(\frac{\partial}{\partial t} - iD_s^*\right) f(t,x) = 0 \quad \text{for all } t > 0,\ x \in M,$$
$$f(0,x) = u(x) \quad \text{for all } x \in M.$$

Since D_s maps basic forms to basic forms, $PD_sP = D_sP$, and $PD_s^*P = PD_s^*$; the form $e^{itD_s}u$ is basic for a given basic form u. Moreover,

$$\frac{\partial}{\partial t}\left(e^{itD_s}u\right) = iD_s\left(e^{itD_s}u\right) = iD_{B,s}\left(e^{itD_s}u\right),$$

so that $e^{itD_{B,s}}u := e^{itD_s}u$ is a solution to the *basic traveling wave equation* with the appropriate initial condition:

$$\left(\frac{\partial}{\partial t} - iD_{B,s}\right)\beta(t,x) = 0 \quad \text{for all } t > 0,\ x \in M,$$
$$\beta(0,x) = u(x) \in \Omega_B^*(M). \tag{3.4}$$

Then $e^{itD_{B,s}}u$ and $e^{-itD_{B,s}}u$ are both solutions of the basic wave equation

$$\left(\frac{\partial^2}{\partial t^2} + D_{B,s}^2\right)\beta(t,x) = 0. \tag{3.5}$$

Because of the results concerning $\frac{\partial}{\partial t} - iD_s$, the solutions to these wave equations exist and are unique with respect to the appropriate initial conditions (both $\beta(0,x)$ and $\frac{\partial}{\partial t}\beta(0,x)$ must be specified for the basic wave equation).

We now prove some analytic results about the operators $D_{B,s}$ and similar operators; we include proofs only where the standard proofs do not translate directly to the basic case. Let $L^p = L^p(\Omega_B^*(M))$ denote the L^p-norm closure of the space of smooth basic forms, and let $W^k = W^k(\Omega_B^*(M))$ denote the closure of the space of these basic forms under the Sobolev $(k,2)$-norm. Let $\|\cdot\|_p$ and $\|\cdot\|_{k,2}$ denote the norms on these spaces.

Lemma 3.4. *Let D be a strongly elliptic, first order operator on $\Omega^*(M)$ that restricts to an operator on $\Omega_B^*(M)$. Suppose that the restriction of D is formally self-adjoint on $\Omega_B^*(M)$. Then, for some $c > 0$, $\|(D^2+1)\alpha\|_{k,2} \geqslant c\|\alpha\|_{k+1,2}$ for every basic form α.*

Proof. We use induction on k. For $k = 0$, we clearly have that

$$\left\|\left(D^2+1\right)\alpha\right\|_{0,2} = \left(\left\langle D^2\alpha, D^2\alpha\right\rangle + 2\langle D\alpha, D\alpha\rangle + \langle\alpha,\alpha\rangle\right)^{1/2} \geqslant \|\alpha\|_{0,2}.$$

Similarly, $\|(D^2+1)\alpha\|_{0,2} \geqslant \sqrt{2}\,\|D\alpha\|_{0,2}$. By ellipticity, there exist $c_1 > 0$ and $c_2 \geqslant 0$ (independent of α) such that $\sqrt{2}\,\|D\alpha\|_{0,2} \geqslant c_1\|\alpha\|_{1,2} - c_2\|\alpha\|_{0,2}$. Thus,

$$\left\|\left(D^2+1\right)\alpha\right\|_{0,2} \geqslant \left(\frac{c_1}{1+c_2}\right)\|\alpha\|_{1,2}.$$

Assume that the conclusion is true for $0 \leqslant k \leqslant m$. Then

$$\begin{aligned}\left\|(D^2+1)\alpha\right\|_{m+1,2} &\geqslant c_1\left\|(D^2+1)^2\alpha\right\|_{m-1,2} \quad \text{by definition of Sobolev norm}\\ &\geqslant c_2\left\|(D^2+1)\alpha\right\|_{m,2} \quad \text{by the induction hypothesis}\\ &\geqslant c_3\|\alpha\|_{m+2,2} - c_4\|\alpha\|_{m+1,2} \quad \text{by ellipticity,}\end{aligned}$$

for positive constants c_1, c_2, and c_3 and for $c_4 \geqslant 0$, all independent of α. Also, it follows from our induction hypothesis that $\|(D^2+1)\alpha\|_{m,2} \geqslant c_5\|\alpha\|_{m+1,2}$ for some $c_5 > 0$. Finally, by the definition of the Sobolev norm, there exists a positive constant c_6 such that

$$\left\|(D^2+1)\alpha\right\|_{m+1,2} \geqslant c_6\left\|(D^2+1)\alpha\right\|_{m+1,2} + \frac{c_6c_4}{c_5}\left\|(D^2+1)\alpha\right\|_{m,2} \geqslant c_6c_3\|\alpha\|_{m+2,2},$$

whence the result follows.

Lemma 3.5. *Let D be as in Lemma* 3.4. *Then there exists a constant $c > 0$ such that $\|(D^2+1)\alpha\|_{k,2} \geqslant c\|\alpha\|_{k+2,2}$ for every basic form α.*

Proof. By ellipticity, there exist $c_1 > 0$ and $c_2 \geqslant 0$ such that $\|(D^2+1)\alpha\|_{k,2} \geqslant c_1\|\alpha\|_{k+2,2} - c_2\|\alpha\|_{k+1,2}$. By Lemma 3.4, there exists $c_3 > 0$ such that $\|(D^2+1)\alpha\|_{k,2} \geqslant c_3\|\alpha\|_{k+1,2}$. Thus

$$\left\|(D^2+1)\alpha\right\|_{k,2} \geqslant \left(\frac{c_1}{1+\frac{c_2}{c_3}}\right)\|\alpha\|_{k+2,2}. \qquad \square$$

Remark 3.6. Observe that if the coefficients of the operator D depend polynomially on a parameter s, then the constants in the inequalities of the lemmas above can be chosen to depend polynomially in s. Also, the results of Lemmas 3.4 and 3.5 extend to much more general situations; the operator D may be any strongly elliptic, first-order operator acting on smooth sections of a vector bundle over a compact manifold such that the restriction of D to a subspace consisting of smooth sections is formally self-adjoint.

Lemma 3.7 (Basic Sobolev embedding theorem). *If $a > \frac{n}{2}$, then $W^a \subset L^\infty$. Also, $W^{a+k} \subset C^k$ for $k > 0$.*

Proof. The standard proof (see [14]) works for the basic case. $\square$

Lemma 3.8. *Define $L_\beta(\alpha) = \int_M (\alpha,\beta)\,\omega_{\mathrm{vol}}$. Then the map $\beta \mapsto L_\beta$ defines a norm-preserving injection of $L^1(\Omega_B^*(M))$ into $(L^\infty(\Omega_B^*(M)))^*$.*

Proof. The standard proof (see [14]) works for the basic case. $\square$

Recall that a subset of a foliation is called *saturated* if it is a union of leaves.

Lemma 3.9. *Let U be a saturated open set in a Riemannian foliation $(M,\mathcal{F})$. Let $P : L^2(\Omega^*(M)) \to L^2(\Omega_B^*(M))$ be the orthogonal projection. Then for every $f \in C^\infty(M)$,*

$$\int_U f\omega_{\mathrm{vol}} = \int_U Pf\omega_{\mathrm{vol}}.$$

Proof. Let $\{\eta_\varepsilon\}$ be a family of smooth basic functions approximating (in L^2) the characteristic function of U as $\varepsilon \to 0$ (see the results of [12] and [9]). Then $\int_M f\eta_\varepsilon\omega_{\mathrm{vol}} = \int_M P(f\eta_\varepsilon)\omega_{\mathrm{vol}} = \int_M (Pf)\eta_\varepsilon\omega_{\mathrm{vol}}$ since η_ε is basic [12]. We then apply the dominated convergence theorem. □

Let $(\cdot\,,\cdot)$ denote the pointwise inner product of forms. The following two results apply to $D_{B,s}$, since $D_{B,s}$ is the restriction of $D_s = d + \delta + \varepsilon + s(i(V) + V^\flat\wedge)$ to basic forms and since $P\varepsilon P = 0$ (see [12]).

Lemma 3.10. *Let D be a first order operator on $\Omega^*(M)$ of the form $D = d + \delta + Z_1 + Z_2$, where*

(1) *D restricts to an operator on $\Omega_B^*(M)$;*
(2) *The operator Z_1 is zeroth order and satisfies $PZ_1P = 0$, where P is the orthogonal projection from $L^2(\Omega^*(M))$ to $L^2(\Omega_B^*(M))$;*
(3) *The operator Z_2 is zeroth order and is formally self-adjoint with respect to the pointwise inner product of forms.*

Let $B(L,R)$ denote the set of points of distance less than R from a fixed leaf closure L; assume that R is sufficiently small to avoid the cut locus of L. Let β be a basic form, and let $\beta_t = \mathrm{e}^{itD}\beta$ be the corresponding solution to the basic traveling wave equation as in (3.4)*. Then, for $0 \leqslant t < R$, the function*

$$f(t) = \int\limits_{B(L,R-t)} (\beta_t, \beta_t)\omega_{\mathrm{vol}}$$

is a decreasing function of t.

Proof. We have

$$\frac{df}{dt} = \int\limits_{B(L,R-t)} \big((iD\beta_t, \beta_t) + (\beta_t, iD\beta_t)\big) - \int\limits_{S(L,R-t)} (\beta_t, \beta_t),$$

where $S(L,r)$ is the set of points of distance R from the leaf closure L. Observe that

$$\begin{aligned}(iD\beta_t, \beta_t) + (\beta_t, iD\beta_t) &= \big(\big(i(d+\delta+Z_1+Z_2)\beta_t, \beta_t\big) + \big(\beta_t, i(d+\delta+Z_1+Z_2)\beta_t\big)\big)\\ &= i\big[\big((d+\delta)\beta_t, \beta_t\big) - \big(\beta_t, (d+\delta)\beta_t\big)\big] + i\big[(Z_1\beta_t, \beta_t) - (\beta_t, Z_1\beta_t)\big],\end{aligned}$$

since Z_2 is formally self-adjoint with respect to $(\cdot\,,\cdot)$. By [14, proof of Proposition 2.9],

$$i\big[\big((d+\delta)\beta_t, \beta_t\big) - \big(\beta_t, (d+\delta)\beta_t\big)\big] = i\delta\omega_t,$$

where ω is the one-form defined by $\omega_t(X) = -(X.\beta_t, \beta_t) := -((X^\flat\wedge - i(X))\beta_t, \beta_t)$. Then

$$\begin{aligned}\int\limits_{B(L,R-t)} \big((iD\beta_t, \beta_t) + (\beta_t, iD\beta_t)\big) &= \int\limits_{B(L,R-t)} i\delta\omega_t + i\big[(Z_1\beta_t, \beta_t) - (\beta_t, Z_1\beta_t)\big]\\ &= \int\limits_{B(L,R-t)} i\delta\omega_t + iP\big[(Z_1P\beta_t, \beta_t) - (\beta_t, Z_1P\beta_t)\big],\end{aligned}$$

which follows from Lemma 3.9 and the fact that β_t is basic. Furthermore, by the results of [12], $P[(Z_1P\beta_t, \beta_t) - (\beta_t, Z_1P\beta_t)] = (PZ_1P\beta_t, \beta_t) - (\beta_t, PZ_1P\beta_t)$. Since $PZ_1P = 0$ by hypothesis, the divergence theorem yields

$$\frac{df}{dt} = i \int\limits_{B(L,R-t)} \delta\omega_t - \int\limits_{S(L,R-t)} (\beta_t, \beta_t) = i \int\limits_{S(L,R-t)} \omega_t(N) - \int\limits_{S(L,R-t)} (\beta_t, \beta_t),$$

where N is the unit vector field normal to $S(L, R-t)$ with orientation chosen compatibly with the choice of orientation of $S(L, R-t)$. Locally $\|\omega_t(N)\|^2 = \|(N.\beta_t, \beta_t)\|^2 \leqslant (\beta_t, \beta_t)(N.\beta_t, N.\beta_t) = (\beta_t, \beta_t)^2$. Since $\frac{df}{dt}$ is real, we conclude that $\frac{df}{dt} \leqslant 0$. □

Proposition 3.11 (Unit propagation speed). *Let D be as in Lemma* 3.10. *Then for any $\beta \in \Omega_B^*(M)$, the support of $\mathrm{e}^{itD}\beta$ lies within a distance $|t|$ of the support of β.*

Proof. One easily checks that such an operator D is formally self-adjoint on the space of basic forms, and $\mathrm{e}^{itD}\mathrm{e}^{\pm isD}\beta = \mathrm{e}^{i(t\pm s)D}\beta$; thus, it is sufficient to prove the result for small positive t. Since β is basic, the support of β and its complement are saturated. Since $(M, \mathcal{F})$ is a Riemannian foliation and since M is compact, for every leaf closure L, there exists a tubular neighborhood and $R > 0$ such that for every leaf closure L_x in that tubular neighborhood, the set $B(L_x, R)$ of points of distance less than R from L_x misses the focal locus and cut locus of L_x. Choose any $x \in M$ that is at a distance R or more from the support of β; let L_x denote the leaf closure containing x. Since $(M, \mathcal{F})$ is Riemannian, the set $B(L_x, R)$ is also disjoint from the support of β. Then

$$\int\limits_{B(L_x,R)} (\beta, \beta) = 0 \geqslant \int\limits_{B(L_x,R-t)} \left(\mathrm{e}^{itD}\beta, \mathrm{e}^{itD}\beta\right) \quad \text{for } 0 < t < R,$$

by Lemma 3.10. Hence $\mathrm{e}^{itD}\beta = 0$ at x for $0 < t < R$.

Lemma 3.12. *Let D be as in Lemma* 3.10, *and let $\phi : \mathbb{R} \to \mathbb{R}$ be a rapidly decreasing even function. Then $\phi(D)$ has a smooth kernel $k(x, y)$ that is basic in each factor and satisfies $k(x, y) = \overline{k(y, x)}$.*

Proof. The operator $\phi(D)$ is a bounded, self-adjoint operator on $L^2(\Omega_B^*)$. By Lemmas 3.4 and 3.5, $\phi(D)$ maps $L^2(\Omega_B^*)$ into W^k for every $k > 0$. By Lemma 3.7, this implies $\phi(D)$ maps $L^2(\Omega_B^*)$ to smooth basic forms. Using the fact that D is formally self-adjoint on $\Omega_B^*(M)$, the standard proofs are easily adapted to show the existence and properties of $k(x, y)$ (see [14, Lemma 5.6]). □

The index of a Fredholm operator is a homotopy invariant; we prove a similar result for the operators $D_{B,s}$ on basic forms. The operators $(\Delta_s')^* = \Delta^j + \delta\varepsilon^* + \varepsilon^*\delta + s(Z')^* + s^2\|V\|^2$ depend smoothly on the parameter s, the vector field V, and the metric. Therefore the kernels $\widetilde{K}_s^{\mathrm{even}}(t, x, y)$ and $\widetilde{K}_s^{\mathrm{odd}}(t, x, y)$ are smooth in these parameters as well. Since the map $\beta \mapsto P\beta$ is continuous, $K_{B,s}^{\mathrm{even}}(t, x, y)$ and $K_{B,s}^{\mathrm{odd}}(t, x, y)$ are continuous in s, V, and the metric. By (3.3) and the fact that the index is an integer, we have shown the following:

Proposition 3.13. $\chi_B(M, \mathcal{F}) = \operatorname{index}(D_{B,s} : \Omega_B^{\mathrm{even}}(M) \to \Omega_B^{\mathrm{odd}}(M))$ *for all $s \in \mathbb{R}$.*

Let ϕ be a smooth, rapidly decreasing function on $[0, \infty)$ with $\phi(0) = 1$. Then $\phi(D_B^2)$ is a trace class operator. Let

$$\mu_j = \operatorname{tr}\left(\phi\left(D_B^2\right)\big|_{L^2(\Omega_B^j(M))}\right).$$

By adapting the argument of [14, Chapter 12] for the standard case, we obtain

Theorem 3.14 (Basic Morse inequalities). *The numbers μ_j and $\beta_j = \dim H_B^j(M, \mathcal{F})$ satisfy the following system of inequalities*:

$$\begin{aligned} \beta_0 &\leqslant \mu_0, \\ \beta_1 - \beta_0 &\leqslant \mu_1 - \mu_0, \\ \beta_2 - \beta_1 + \beta_0 &\leqslant \mu_2 - \mu_1 + \mu_0, \end{aligned}$$

etc., and the equality

$$\chi_B(M, \mathcal{F}) = \operatorname{index}\bigl(D_B : \Omega_B^{\mathrm{even}}(M) \to \Omega_B^{\mathrm{odd}}(M)\bigr) = \sum_{j=0}^{q} (-1)^j \beta_j = \sum_{j=0}^{q} (-1)^j \mu_j.$$

Proof. The proof is identical to the proof of the standard case in [14, Chapter 12], replacing d with d_B and the standard Hodge theorem with the basic Hodge theorem. □

Let

$$\mu_s^{\mathrm{even}} = \operatorname{tr}\bigl(\phi\bigl(D_{B,s}^2\bigr)\big|_{L^2(\Omega_B^{\mathrm{even}}(M))}\bigr), \qquad \mu_s^{\mathrm{odd}} = \operatorname{tr}\bigl(\phi\bigl(D_{B,s}^2\bigr)\big|_{L^2(\Omega_B^{\mathrm{odd}}(M))}\bigr).$$

Combining the proof of the standard Morse inequalities with Proposition 3.13, we have the following result.

Proposition 3.15. $\chi_B(M, \mathcal{F}) = \mu_s^{\mathrm{even}} - \mu_s^{\mathrm{odd}}$ *for all* $s \in \mathbb{R}$.

Fix a number $\rho > 0$, and choose a positive, even Schwartz function ϕ with $\phi(0) = 1$ and such that the Fourier transform $\hat{\phi}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-ix\xi} \phi(x)\, dx$ is supported in the interval $[-\rho, \rho]$. Since ϕ is even, $\phi(D_{B,s})$ makes sense as a smoothing operator, and the basic Euler characteristic satisfies Proposition 3.15 with

$$\mu_s^{\mathrm{even}} = \operatorname{tr}\bigl(\phi(D_{B,s})|_{L^2(\Omega_B^{\mathrm{even}}(M))}\bigr) \quad \text{and} \quad \mu_s^{\mathrm{odd}} = \operatorname{tr}\bigl(\phi(D_{B,s})|_{L^2(\Omega_B^{\mathrm{odd}}(M))}\bigr).$$

Let $\operatorname{Crit}(V)$ be the (finite) union of critical leaf closures of V in M.

Lemma 3.16. *On the complement of a 2ρ-neighborhood of* $\operatorname{Crit}(V)$, *the basic kernel of $\phi(D_{B,s})$ satisfies $k_{B,s}(x, y) \to 0$ uniformly as $s \to \infty$.*

Proof. This proof is the same as [14, Lemma 12.10], but we include some details here because of subtleties. Choose a constant C so that $\|V(x)\| \geqslant C > 0$ for all x in the complement of a ρ-neighborhood of $\operatorname{Crit}(V)$. By formula (3.2) and the remarks following that equation, $\langle D_{B,s}^2 \beta, \beta\rangle \geqslant \frac{C^2 s^2}{2} \langle \beta, \beta\rangle$ for every $\beta \in \Omega_B^*(M)$ that is supported on the complement of such a neighborhood and for sufficiently large s. Let $\mathcal{H}$ denote the Hilbert space of L^2 basic forms that vanish on a ρ-neighborhood of $\operatorname{Crit}(V)$. Then $D_{B,s}^2$ is a positive symmetric operator on a dense subset of $\mathcal{H}$, so it extends to a self-adjoint operator A on $\mathcal{H}$ satisfying the same inequality above.

Let ω be a basic form supported on the complement of a 2ρ-neighborhood of $\operatorname{Crit}(V)$, and let

$$\omega_t = \cos(t D_{B,s})\omega = \frac{1}{2}\bigl(\mathrm{e}^{itD_{B,s}} + \mathrm{e}^{-itD_{B,s}}\bigr)\omega,$$

which is a solution to the generalized wave equation (3.5) corresponding to the operator Δ'_s with initial conditions $\omega_0 = \omega$, $\frac{\partial}{\partial t}\omega_0 = 0$. The family of forms ω_t is the unique solution to this generalized wave equation as well, by the statements before and after (3.5). Note that the formula above implies that ω_t is basic.

By the unit propagation speed property of the basic wave equation (Proposition 3.11), ω_t is identically zero on the ρ-neighborhood of $\mathrm{Crit}(V)$ if $|t| < \rho$. This implies that $D^2_{B,s}\omega_t = A\omega_t$ for $|t| < \rho$, so that ω_t is the unique solution to the system

$$\frac{\partial^2}{\partial t^2}\omega_t + A\omega_t = 0; \quad \omega_0 = \omega, \quad \frac{\partial}{\partial t}\omega_0 = 0.$$

We may therefore write $\omega_t = \cos(t\sqrt{A})\omega$.

Let ϕ be a real-valued function with the following properties:

(1) ϕ is a positive even Schwartz function;
(2) $\phi(0) = 1$;
(3) the Fourier transform $\hat{\phi}(t)$ is supported in the interval $[-\rho, \rho]$.

For each nonnegative integer m, define ϕ_m by the formula

$$\phi_m(\lambda) = \left(1 + \lambda^2\right)^{2m}\phi(\lambda);$$

note that each ϕ_m satisfies (1)–(3).

For a basic form ω that is supported on the complement of the 2ρ-neighborhood of $\mathrm{Crit}(V)$,

$$\begin{aligned}
\phi_m(D_{B,s})\omega &= \frac{1}{2\pi}\int_{-\rho}^{\rho} \hat{\phi}_m(t)\left(\mathrm{e}^{itD_{B,s}}\omega\right) dt \\
&= \frac{1}{2\pi}\int_{-\rho}^{\rho} \hat{\phi}_m(t)\left(\cos(tD_{B,s})\omega\right) dt \quad \text{since } \hat{\phi}_m \text{ is even} \\
&= \frac{1}{2\pi}\int_{-\rho}^{\rho} \hat{\phi}_m(t)\left(\cos\left(t\sqrt{A}\right)\omega\right) dt \\
&= \phi_m\left(\sqrt{A}\right)\omega.
\end{aligned} \tag{3.6}$$

The operator $\sqrt{A}$ is positive and has operator norm is bounded below by $Cs/\sqrt{2}$ for s sufficiently large. Thus, the operator norm of $\phi_m(\sqrt{A})$ (as an operator from $\mathcal{H}$ to itself) is bounded above by

$$c_m(s) = \sup\left\{\left|\phi_m(\lambda)\right|: \lambda \geqslant \frac{Cs}{\sqrt{2}}\right\}.$$

It is clear that $c_m(s)$ is rapidly decreasing as $s \to \infty$. By (3.6),

$$\left\|\phi_m(D_{B,s})\omega\right\|_2 \leqslant c_m(s)\|\omega\|_2$$

for every basic form ω supported on the complement of a 2ρ-neighborhood of $\mathrm{Crit}(V)$.

Next, let $L^p = L^p(\Omega_B^*(M))$ denote the L^p-norm closure of the space of smooth basic forms, and let $W^k = W^k(\Omega_B^*(M))$ denote the closure of the space of such basic forms under the Sobolev $(k,2)$-norm. By the basic elliptic estimates (Lemma 3.5), the operator norm of $(1 + D_{B,s}^2)^{-1} : W^k \to W^{k+2}$ is bounded by a polynomial in s. The basic version of the Sobolev embedding theorem (Lemma 3.7) implies that $(1 + D_{B,s}^2)^{-k} : L^2 \to L^\infty$ is a bounded map whose operator norm is bounded by a polynomial in s if $k > \frac{n}{4}$. Using basic duality (Lemma 3.8) and essential self-adjointness of $D_{B,s}$, we see that $(1 + D_{B,s}^2)^{-k} : L^1 \to L^2$ is also a bounded map whose operator norm is bounded by a polynomial in s whenever $k > \frac{n}{4}$. Note that all of the statements above hold for the operator A as well as for $D_{B,s}^2$. Now, given a basic form ω supported on the complement of a 2ρ-neighborhood of $\mathrm{Crit}(V)$ and $k > \frac{n}{4}$,

$$\begin{aligned}
\left\|\phi(D_{B,s})\omega\right\|_\infty &= \left\|\phi\big(\sqrt{A}\big)\omega\right\|_\infty \left\|(1+A)^{-k}(1+A)^k\phi\big(\sqrt{A}\big)(1+A)^k(1+A)^{-k}\omega\right\|_\infty \\
&\leqslant \left\|(1+A)^{-k}\right\|_{L^2\to L^\infty} c_k(s)\left\|(1+A)^{-k}\omega\right\|_2 \\
&\leqslant \left\|(1+A)^{-k}\right\|_{L^2\to L^\infty} c_k(s)\left\|(1+A)^{-k}\right\|_{L^2\to L^1}\|\omega\|_1 \\
&\leqslant p(s)c_k(s)\|\omega\|_1,
\end{aligned}$$

where $p(s)$ is a polynomial in s. Next, since ϕ is rapidly decreasing, $\phi(D_{B,s})$ has a continuous basic kernel $k_{B,s}(x,y)$ (Lemma 3.12), and we have the inequality

$$\left\|k_{B,s}(x,\cdot)\right\|_\infty \leqslant \sup_{\int\|\omega\|=1,\, x\in M}\left\|\int_M k_{B,s}(x,y)\omega(y)\omega_{\mathrm{vol}}(y)\right\| \leqslant p(s)c_k(s)$$

from the above. Thus, $k_{B,s}(x,y) \to 0$ uniformly as $s \to \infty$. □

Let $(M,\mathcal{F})$ be a Riemannian foliation, let V be a $\mathcal{F}$-nondegenerate basic vector field, let L be a leaf closure of $\mathcal{F}$, and let $\mathcal{O}_L = \mathcal{O}_L(V)$ denote the orientation line bundle of V at L. We denote by $\Omega^*(L,\mathcal{F},\mathcal{O}_L)$ the space of differential forms on L with values in $\mathcal{O}_L$; there is a well-defined differential d on this space that is simply the exterior derivative when $\mathcal{O}_L$ is trivial (see [2, Section I.7]). Locally an element of $\Omega^*(L,\mathcal{F},\mathcal{O}_L)$ can be written as a sum of tensors $\omega\otimes s$, where ω is an ordinary differential form on M and s is a smooth section of $\mathcal{O}_L$. Let X be a vector field on L. Then we can locally define interior multiplication $i(X)$ on $\Omega^*(L,\mathcal{F},\mathcal{O}_L)$ by decreeing that $i(X)(\omega\otimes s) = (i(X)\omega)\otimes s$ and extending linearly. It is straightforward to check that this definition of interior product is independent of the trivialization of $\mathcal{O}_L(V)$, and so we may therefore make the following definitions:

Definition 3.17. Let M, $\mathcal{F}$, V, and L have the properties listed in the previous paragraph. The space of basic differential forms with values in $\mathcal{O}_L$ is denoted $\Omega_B^*(L,\mathcal{F},\mathcal{O}_L)$, and is defined to be the subcomplex of forms α in $\Omega^*(L,\mathcal{F},\mathcal{O}_L)$ with the property that $i(X)\alpha = 0$ and $i(X)\,d\alpha = 0$ for every vector field X on L tangent to the leaves of $\mathcal{F}$ restricted to L. The cohomology of this subcomplex is denoted $H_B^*(L,\mathcal{F},\mathcal{O}_L)$ and called the basic de Rham cohomology of L with values in $\mathcal{O}_L$. Finally, the basic Euler characteristic of L with values in $\mathcal{O}_L$ is defined by the formula $\chi_B(L,\mathcal{F},\mathcal{O}_L) = \sum_k(-1)^k \dim H_B^k(L,\mathcal{F},\mathcal{O}_L)$.

Theorem 3.18 (Basic Hopf Index theorem). *Let M, $\mathcal{F}$, and V be as in Definition* 3.17. *For each critical leaf closure L, let* $\mathrm{ind}_L(V)$ *be the index of V at L. Then*

$$\chi_B(M,\mathcal{F}) = \sum_{L\text{ critical}} \mathrm{ind}_L(V)\chi_B(L,\mathcal{F},\mathcal{O}_L).$$

Proof. Without loss of generality, we may assume that V is a basic normal vector field (otherwise, project to $N\mathcal{F}$). By Proposition 3.15, we have that $\chi_B(M,\mathcal{F}) = \lim_{s\to\infty}(\mu_s^{\text{even}} - \mu_s^{\text{odd}})$, which may be obtained by integrating the traces of the corresponding kernels of $\phi(D_{B,s}^2)|_{L^2(\Omega_B^{\text{even}}(M))}$ and $\phi(D_{B,s}^2)|_{L^2(\Omega_B^{\text{odd}}(M))}$. By Proposition 3.16, the kernels of these operators go to zero uniformly on the complement of a fixed but arbitrarily small neighborhood of the critical leaf closures of V. For each critical leaf closure L, let ψ_L be a smooth, radial, basic function that is identically 1 in a tubular neighborhood of radius 2ρ around L and supported within a tubular neighborhood of radius 3ρ (assume that we have chosen ρ small enough so that this is possible for each L). Then we have that

$$\chi_B(M,\mathcal{F}) = \sum_{L \text{ critical}} \lim_{s\to\infty} \operatorname{tr}\bigl(\psi_L\phi\bigl(D_{B,s}^2\bigr)\big|_{L^2(\Omega_B^{\text{even}}(M))}\bigr) - \operatorname{tr}\bigl(\psi_L\phi\bigl(D_{B,s}^2\bigr)\big|_{L^2(\Omega_B^{\text{odd}}(M))}\bigr).$$

We now use Lemma 3.3 to observe that V may be deformed without changing $\operatorname{ind}_L(V)$ or $\mathcal{O}_L$ so that within a tubular neighborhood of radius 4ρ around L, $V = \nabla f$ for a basic function f, such that if $\operatorname{ind}_L(V) = +1$ then f has even Morse index and if $\operatorname{ind}_L(V) = -1$ then f has odd Morse index; again we possibly decrease ρ so that the conclusion of this proposition holds. A unit propagation speed argument shows that the traces are independent of the choice of V with those properties, and thus we may calculate the contributions from each tubular neighborhood as if $V = \nabla f$. By the results of [1], if D_L^2 is the basic Laplacian on L with coefficients in $\mathcal{O}_L$, then

$$\lim_{s\to\infty} \operatorname{tr}\bigl(\psi_L\phi\bigl(D_{B,s}^2\bigr)\big|_{L^2(\Omega_B^{\text{even}}(M))}\bigr) = \operatorname{tr}\Bigl(\phi\bigl(D_L^2\bigr)\Big|_{L^2\bigl(\Omega_B^{\text{even}+\frac{\operatorname{ind}_L(V)-1}{2}}(L,\mathcal{F},\mathcal{O}_L)\bigr)}\Bigr),$$

with the analogous result for the odd case. Thus,

$$\begin{aligned}
\chi_B(M,\mathcal{F}) &= \sum_{L \text{ critical}} \operatorname{tr}\Bigl(\phi\bigl(D_L^2\bigr)\Big|_{L^2\bigl(\Omega_B^{\text{even}+\frac{\operatorname{ind}_L(V)-1}{2}}(L,\mathcal{F},\mathcal{O}_L)\bigr)}\Bigr) - \operatorname{tr}\Bigl(\phi\bigl(D_L^2\bigr)\Big|_{L^2\bigl(\Omega_B^{\text{odd}+\frac{\operatorname{ind}_L(V)-1}{2}}(L,\mathcal{F},\mathcal{O}_L)\bigr)}\Bigr) \\
&= \sum_{L \text{ critical}} \dim H_B^{\text{even}+\frac{\operatorname{ind}_L(V)-1}{2}}(L,\mathcal{F},\mathcal{O}_L) - \dim H_B^{\text{odd}+\frac{\operatorname{ind}_L(V)-1}{2}}(L,\mathcal{F},\mathcal{O}_L) \\
&= \sum_{\substack{L \text{ critical} \\ \operatorname{ind}_L(V)=+1}} \chi_B(L,\mathcal{F},\mathcal{O}_L) - \sum_{\substack{L \text{ critical} \\ \operatorname{ind}_L(V)=-1}} \chi_B(L,\mathcal{F},\mathcal{O}_L). \qquad \square
\end{aligned}$$

Remark 3.19. If all of the bundles $\mathcal{O}_L$ are trivial and the critical leaf closures are leaves, then the formula simplifies to

$$\chi_B(M,\mathcal{F}) = \sum_{L \text{ critical}} \operatorname{ind}_L(V),$$

a formula which has the precise form of the ordinary Hopf Index Theorem.

Corollary 3.20. *Suppose that there exists a basic vector field on a Riemannian foliation $(M,\mathcal{F})$ that is nowhere tangent to $\mathcal{F}$. Then $\chi_B(M,\mathcal{F}) = 0$.*

Corollary 3.21. *Suppose that there exists a basic vector field on a Riemannian foliation $(M,\mathcal{F})$ that is nowhere tangent to $\mathcal{F}$ and that the codimension of $\mathcal{F}$ is less than 3. Then M has infinite fundamental group.*

Proof. The hypotheses imply

$$\begin{aligned}\chi_B(M,\mathcal{F}) &= \dim H_B^0(M,\mathcal{F}) - \dim H_B^1(M,\mathcal{F}) + \dim H_B^2(M,\mathcal{F}) \\ &= 0.\end{aligned}$$

Since $\dim H_B^0(M,\mathcal{F}) \geqslant 1$, $\dim H_B^1(M,\mathcal{F}) \geqslant 1$. Since $H_B^1(M,\mathcal{F})$ injects into $H^1(M)$ [16, Proposition 4.1], Poincaré duality and the Universal Coefficient Theorem imply that rank $H_1(M) \geqslant 1$. The result follows. □

We now illustrate Theorem 3.18 in the following examples. In each of these examples, there does not exist a basic vector field that is nowhere tangent to the foliation.

Example 3.22. Consider the sphere S^2 in spherical coordinates $(\theta,\varphi) \in [0,2\pi] \times [0,\pi]$. Let α be a fixed irrational multiple of π, and let $M = \mathbb{R} \times S^2/\sim$, where $(t,\theta,\varphi) \sim (t+1,\theta+\alpha,\varphi)$. The t-parameter curves make M into a codimension-2 foliation $\mathcal{F}$, which is Riemannian with respect to the standard metric. Observe that the leaf closures are level sets where φ is constant. Consider the vector field

$$V = (\cos\varphi)\partial_\theta + (\sin\varphi\cos\varphi)\partial_\varphi.$$

This vector field is invariant under rotations in θ and is smooth on S^2, since it is the restriction of

$$\left(xz^2 - yz\right)\partial_x + \left(yz^2 + xz\right)\partial_y + \left(-x^2z - y^2z\right)\partial_z$$

on $\mathbb{R}^3$; therefore, it is a smooth basic vector field on $(M,\mathcal{F})$. The critical leaf closures for this vector field correspond to the poles ($z = \pm 1$) and the equator ($\varphi = \pi/2$). At the north pole leaf L_1 ($z = 1$), the matrix for V_{L_1} is $\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ in $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)$ coordinates, so that $\mathrm{ind}_{L_1}(V) = 1$. At the south pole L_{-1} ($z = -1$), the matrix for $V_{L_{-1}}$ is $\left(\begin{smallmatrix}-1 & 1\\ -1 & -1\end{smallmatrix}\right)$ in $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)$ coordinates, so that $\mathrm{ind}_{L_{-1}}(V) = 1$. For these leaf closures, we have $\chi_B(L_{\pm1},\mathcal{F},\mathcal{O}_{L_{\pm1}}) = 1$ since $\mathcal{O}_{L_{\pm1}}$ and $\mathcal{F}$ are trivial. At the equator L_0, V_{L_0} is multiplication by -1 on each 1-dimensional normal space to L_0, so that $\mathrm{ind}_{L_0}(V) = -1$. The orientation bundle is the trivial conormal bundle. Next, observe that this leaf closure is a flat torus, on which the foliation restricts to be the irrational flow. Since the vector field ∂_θ is basic, nonsingular, and orthogonal to the foliation on this torus,

$$\chi_B(L_0,\mathcal{F},\mathcal{O}_{L_0}) = \chi_B(L_0,\mathcal{F}) = 0$$

by Corollary 3.20. By Theorem 3.18, we conclude that

$$\chi_B(M,\mathcal{F}) = \sum_{\substack{L \text{ critical}\\ \mathrm{ind}_L(V)=+1}} \chi_B(L,\mathcal{F},\mathcal{O}_L) - \sum_{\substack{L \text{ critical}\\ \mathrm{ind}_L(V)=-1}} \chi_B(L,\mathcal{F},\mathcal{O}_L) = (1+1) - (0) = 2.$$

In this example, one may independently verify that $\dim H_B^k(M,\mathcal{F}) = 1$ for $k = 0$ or $k = 2$, since M is a transversally oriented, taut, codimension-2 Riemannian foliation (see [16]). Also, every closed basic one-form can be written as $g(\varphi)\,d\varphi$ for a smooth function g such that $\frac{\partial^m g}{\partial\varphi^m} = 0$ for even m at $\varphi = 0$ or π. Every exact basic one-form is the differential of a smooth function h such that $\frac{\partial^m h}{\partial\varphi^m} = 0$ for odd m at $\varphi = 0$ or π; thus, $\dim H_B^1(M,\mathcal{F}) = 0$. We have therefore directly computed that $\chi_B(M,\mathcal{F}) = 1 - 0 + 1 = 2$. Observe that Corollary 3.20 implies that there does not exist a basic vector field on $(M,\mathcal{F})$ that is nowhere tangent to the leaves.

Example 3.23. Let X be any smooth, closed manifold whose fundamental group is the free product $\mathbb{Z}*\mathbb{Z}$ (such as the connected sum of two copies of $S^2 \times S^1$); choose a metric so that the volume of X is one. Let $\widetilde{X}$ denote the universal cover of X. Suspend an action of the fundamental group on the torus to get the manifold $Y = \widetilde{X} \times S^1 \times S^1/\sim$, where the equivalence relation is defined as follows. Let a and b be the generators of $\pi_1(X)$, which act on the right by isometries on $\widetilde{X}$. Let α be an irrational multiple of 2π, and let

$$\begin{aligned}&(x,\theta_1,\theta_2) \sim (xa, \theta_1+\alpha, \theta_2), \quad \text{and}\\ &(x,\theta_1,\theta_2) \sim (xb, 2\pi-\theta_1, 2\pi-\theta_2).\end{aligned}$$

These actions on the torus (with the flat metric) are orientation-preserving isometries, so that the x-parameter (immersed) submanifolds form a transversally oriented, Riemannian foliation $(Y,\mathcal{F})$. The leaf closures are parametrized by $\theta_2 \in [0,\pi]$; observe that the leaf closures corresponding to $\theta_2 = 0$ and $\theta_2 = \pi$ are not transversally oriented, even though $\mathcal{F}$ is.

We first calculate $\chi_B(Y,\mathcal{F})$ directly. As in the previous example, we observe that $\dim H_B^k(Y,\mathcal{F}) = 1$ for $k=0$ or $k=2$, since M is a transversally oriented, taut, codimension-2 Riemannian foliation. Each closed, basic one-form can be written as $g(\theta_2)\,d\theta_2$, where g is a smooth function on S^1 such that $g(2\pi-\theta_2) = -g(\theta_2)$. Since this is the differential of the well-defined basic function $f(\theta_2) = \int_0^{\theta_2} g(t)\,dt$, we see that $\dim H_B^1(Y,\mathcal{F}) = 0$. Therefore, $\chi_B(Y,\mathcal{F}) = 1-0+1 = 2$.

Next, consider the basic vector field $W = \sin(\theta_2)\partial_{\theta_2}$. This vector field is singular at the leaf closures $\theta_2 = 0$ and $\theta_2 = \pi$ (both of codimension 1), and it is orthogonal to the leaves everywhere. The index of W at $\theta_2 = 0$ is 1, and its index at $\theta_2 = \pi$ is -1. The polar decomposition of the linearization of W at $\theta_2 = 0$ is simply $1*1$, so there are no negative eigenvectors of the orthogonal part. Thus, the orientation line bundle is trivial at $\theta_2 = 0$. Therefore,

$$\chi_B\big(\{\theta_2=0\},\mathcal{F},\mathcal{O}_L\big) = \chi_B\big(\{\theta_2=0\},\mathcal{F}\big) = 1-0 = 1;$$

note that there are no closed one-forms on this leaf closure, because it is not transversally oriented. The polar decomposition of the linearization of W at $\theta_2 = \pi$ is $1*(-1)$, and the orientation line bundle $\mathcal{O}_\pi$ is simply the normal bundle to the leaf closure. This bundle is nontrivial and has no basic sections, so $\dim H_B^0(\{\theta_2=0\},\mathcal{F},\mathcal{O}_L) = 0$. On the other hand, let s denote a basic section of the pullback of $\mathcal{O}_\pi$ via the lift of the leaf closure $\theta_2 = \pi$ to $\widetilde{X} \times S^1 \times S^1$; such a section exists because the pullback of $\mathcal{O}_\pi$ is trivial. The basic one-form $d\theta_1 \otimes s$ is closed and descends to a basic one-form on Y with values in $\mathcal{O}_\pi$, because the orientation-reversing action of b changes the sign of both $d\theta_1$ and s. It follows that $\dim H_B^1(\{\theta_2=0\},\mathcal{F},\mathcal{O}_L) = 1$. Thus,

$$\chi_B\big(\{\theta_2=0\},\mathcal{F},\mathcal{O}_L\big) = 0-1 = -1.$$

By Theorem 3.18,

$$\chi_B(Y,\mathcal{F}) = \sum_{\substack{L \text{ critical}\\ \operatorname{ind}_L(V)=+1}} \chi_B(L,\mathcal{F},\mathcal{O}_L) - \sum_{\substack{L \text{ critical}\\ \operatorname{ind}_L(V)=-1}} \chi_B(L,\mathcal{F},\mathcal{O}_L) = (1)-(-1) = 2.$$

These calculations show that there does not exist a basic vector field on $(Y,\mathcal{F})$ that is nowhere tangent to the foliation; in fact, any $\mathcal{F}$-nondegenerate basic vector field on $(Y,\mathcal{F})$ must be tangent to the foliation on at least two distinct leaf closures.

Example 3.24. Let N be a smooth, four-dimensional closed manifold with finite fundamental group. Let $(M,\mathcal{F})$ be any foliation that is obtained by suspending a discrete subgroup Γ of a compact Lie group of diffeomorphisms of N. That is, choose a manifold X along with a surjective homomorphism $\phi:\pi_1(X)\to\Gamma$, and let $M=\widetilde{X}\times N/\pi_1(X)$, where $\pi_1(X)$ acts on the universal cover $\widetilde{X}$ by deck transformations and on N via ϕ. The foliation $\mathcal{F}$ is locally given by the $\widetilde{X}$-parameter submanifolds. Choose metrics for X and N; by averaging the metric on N over the Lie group of diffeomorphisms, we may and do assume that $\pi_1(X)$ acts on N by isometries. The metric on M defined locally as the product of these metrics is bundle-like for the foliation $\mathcal{F}$. Furthermore, this foliation is taut, so the standard form of Poincaré duality holds for basic cohomology (see [16]). The basic forms of $(M,\mathcal{F})$ are given by forms on N that are invariant under the discrete group of isometries, and the basic cohomology is isomorphic to the cohomology of invariant forms on N. Since $\pi_1(N)$ is finite, $H_B^1(M,\mathcal{F})$ is trivial. We conclude that

$$\begin{aligned}\chi_B(M,\mathcal{F})&=\sum_{j=0}^{4}(-1)^j\dim H_B^j(M,\mathcal{F})\\&=\dim H_B^0(M,\mathcal{F})+\dim H_B^2(M,\mathcal{F})+\dim H_B^4(M,\mathcal{F})\\&=2+\dim H_B^2(M,\mathcal{F})\geqslant 2.\end{aligned}$$

Theorem 3.18 implies that every $\mathcal{F}$-nondegenerate basic vector field on $(M,\mathcal{F})$ must have at least two distinct critical leaf closures.

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