THE MATHEMATICS OF APPORTIONMENT

EFTON PARK[†]

An important aspect of any representative democracy is the manner in which its legislative bodies are chosen. This process involves many components, and a fundamental question is this: how does one fairly apportion representatives in a legislature? *A priori*, this seems to be a question that is easily answered, but in fact apportionment is a complicated problem, both mathematically and politically. The Founding Fathers recognized both the importance and difficulty of the apportionment problem, and several of them, including Thomas Jefferson and Alexander Hamilton, proposed apportionment methods. George Washington exercised the first presidential veto in history by rejecting Hamilton's method of apportionment.[1]

In this Article, I will discuss the various methods of apportionment that have been used in the U.S. House of Representatives, and I will particularly focus on the unusual mathematical problems and paradoxes that arise in apportionment. These problems are more than just an academic curiosity—the legitimacy of apportionment is based primarily on the perception (of both Congress and the voting public) that the apportionment method in use is fair. However, as we shall see, every method of apportionment has particular flaws and biases. Also, the issue of what constitutes a "fair" apportionment method is quite debatable. For these reasons and others, apportionment has been a very contentious issue at many times during the history of the United States.

This Article will discuss some of the many interesting aspects of apportionment, and I will illustrate the features and drawbacks of various apportionment methods by considering some specific numerical examples. For more information on both the history and mathematics of apportionment, the reader is urged to consult the excellent monograph on this subject written by Balinski and Young.[2]

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1. John C. Fitzpatrick, ed, 32 *The Writings of George Washington* from the Original Manuscript Sources, 1745-1799 16-7 (Government Printing Office 1931-1944).

2. M.L. Balinski and H. Peyton Young, *Fair Representation* (Yale 1982).

THE APPORTIONMENT PROBLEM

Let us begin by precisely stating the problem.

Given states s_1, s_2, . . ., s_n and a house of size h, choose whole numbers a_1, a_2, . . ., a_n so that $a_1 + a_2 + \cdots + a_n = h$.

For each state s_k, the number a_k is its *apportionment*. In most cases, there are additional restrictions placed on the apportionments. For example, if we wish to apportion the U.S. House of Representatives, we are bound by Article I, Section 2 of the U.S. Constitution: "The Number of Representatives shall not exceed one for every thirty thousand, but each State shall have at least one Representative." In addition, it is desirable that a state's apportionment be roughly proportional to the population of the state. To put this in mathematical terms, let p_1, p_2, . . ., p_n be the populations of the states s_1, s_2, . . ., s_n, respectively, and let $p = p_1 + p_2 + \cdots + p_n$ be the total population. Then for each state s_k, its *quota* q_k is the number $p_k h/p$. Ideally, each state's apportionment should be equal to its quota. The problem is that the quota is almost always a number with a fractional part, whereas most legislative bodies do not allow fractional numbers of representatives. Therefore, we need a method of rounding the quotas to whole numbers. As we shall see, this ostensibly simple rounding issue is in reality quite problematic.

METHODS OF APPORTIONMENT

There are many methods of apportionment that have been suggested (and in fact, there are an infinite number of different apportionment methods). In this Article, I consider the following methods: Jefferson's method, attributable to Thomas Jefferson and used from 1791 until 1830; Webster's method, devised by Daniel Webster and used in the 1840, 1910, and 1930 apportionments; Hill's method, attributable to Joseph A. Hill[3], used from 1940 to present; and Hamilton's method, proposed by Alexander Hamilton, used from 1850 to 1900.[4] A fifth method, that of John Quincy Adams, was considered by Congress but never adopted.

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3. Hill was a statistician with the U.S. Census Bureau. This method is sometimes called the Huntington method or the Huntington-Hill method, for Edward V. Huntington, professor of mechanics and mathematics at Harvard. Huntington was a classmate of Hill's when both were students at Harvard and was an influential proponent of the method. See Balinski and Young, *Fair Representation* at 50 (cited in note 2).

4. The observant reader will notice that the apportionment method for the 1920 census data is not present. This is because the apportionment debate in Congress was so divisive throughout the 1920's that Congress never agreed on an apportionment method, so the U.S. House was not reapportioned until after

All of these apportionment schemes, except for Hamilton's, are examples of *divisor methods.*

> *General Divisor Method.* Choose the house size h, and find a number d, called a *divisor*, so that $rnd(p_1/d) + rnd(p_2/d) + \ldots + rnd(p_n/d) = h$, where "rnd" is some rounding rule. Then for each state s_k, its apportionment is $rnd(p_k/d)$.

Note that such a divisor d can always be chosen, because the left hand side of the equation can be made arbitrarily large or small by making d smaller or larger, respectively. Also observe that typically there is not a unique value for d; usually there is a range of values that will work. However, all divisors that yield a given house size will produce the same apportionment.

Tables 1 and 2 describe the various rounding rules and the divisor methods associated with them.

Table 1: Rounding Rules

Name	Description	Examples
floor(x)	nearest whole number less than or equal to x	floor(2.2) = 2 floor(7.9) = 7
ceiling(x)	nearest whole number greater than or equal to x	ceiling(2.2) = 3 ceiling(7.9) = 8
near(x)	whole number nearest to x	near(2.2) = 2 near(7.9) = 8
gmr(x)	round up or down according to whether x is greater than or less than the geometric mean of floor(x) and ceiling(x), respectively (the geometric mean of two numbers is the square root of their product).	the geometric mean of 2 and 3 is approx. 2.449, so for example gmr(2.44) = 2 gmr(2.45) = 3

Table 2: Divisor Methods

Name of Method	Rounding Rule
Jefferson	floor

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the 1930 census had been taken. For more information, see Balinski and Young, *Fair Representation* at 53 (cited in note 2).

Adams   ceiling

Webster   near

Hill   gmr

Finally, we define Hamilton's method.

Hamilton's Method. Choose the house size h, compute the quotas q_1, q_2, . . ., q_n, and for each state s_k, assign it floor(q_k) seats. Then distribute the remaining seats to those states for which the fractional part of their quota is the largest.

To better understand these apportionment methods, it is instructive to examine an example involving a fictitious country with eight states.

Table 3: A Comparison of the Apportionment Methods

State	Population	Quota	Jefferson	Adams	Webster	Hill	Hamilton
State A	1,908,578	40.705	42	39	41	40	41
State B	1,366,072	29.135	30	28	29	29	29
State C	651,832	13.902	14	14	14	14	14
State D	250,657	5.346	5	6	5	5	5
State E	163,904	3.496	3	4	4	4	3
State F	157,147	3.352	3	4	3	3	3
State G	120,419	2.568	2	3	3	3	3
State H	70,173	1.497	1	2	1	2	2
Total	4,688,782	100	100	100	100	100	100
Divisor[5]			44,600	49,000	46,800	47,300	--

Note that each of the methods gives a different apportionment in this example, so these five methods are truly distinct.

Observe that State A's quota is 40.705. In the interest of fairness, it would seem that State A should receive either forty or forty-one representatives. However, Jefferson's method assigns State A forty-two representatives, while Adams' method assigns only thirty-nine representatives to State A. These are examples of *violation of quota*; by definition, violation of quota occurs whenever a state's apportionment differs from its quota by more than one seat.

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5. The divisors are computed by trial and error; there are no formulas for them.

Violation of quota is very likely to occur with either Jefferson's or Adams' method and somewhat less likely with Webster's or Hill's method, but can occur in any divisor method; I will have more to say about this later. Quota violation can never occur with Hamilton's method, but as we shall see in the next section, Hamilton's method suffers from other problems.

The data in Table 3 may lead one to suspect that Jefferson's method favors states with large populations[6], while Adams' method favors the small states; this is generally true. Hill's method also favors small states, but not to the extent that Adams' method does. On the other hand, Hamilton's method has a slight bias toward large states.[7] Finally, Webster's method generally treats large and small states equally.[8] Now, an examination of Table 3 shows that Hamilton's method assigns two representatives to State H, while Webster's method assigns only one representative to that state. This seems to contradict the assertion that Hamilton's method favors large states. However, to say that an apportionment method is biased toward large states (for example) means that over many elections, the method *systematically* favors those states; the results of any one apportionment are not sufficient to show bias.

APPORTIONMENT PARADOXES

The violation of quota problem that arises in divisor methods is certainly undesirable. However, far stranger things can occur in seemingly reasonable apportionment methods. Of these, perhaps the strangest is the *Alabama paradox*, whose namesake was almost its victim after the 1880 census.[9] I demonstrate the paradox by considering State H from Table 3 and examining its apportionment under Hamilton's method as the size of the house increases:

Table 4: Apportionment for State H under Hamilton's Method

House Size	100	105	107	108	111	113	115
Apportionment	2	1	2	1	2	1	2

Table 4 demonstrates that it is possible under Hamilton's method for a state's apportionment to decrease when the house size increases, and in fact this can happen multiple times. The state of Maine was adversely affected in this way

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6. I will leave it to the reader to ponder the significance of the fact that Jefferson was from Virginia, and that at the time Jefferson proposed his method, Virginia was the most populous state in the union.

7. Balinski and Young, *Fair Representation* at 83.

8. Id at 76.

9. Congressional Record, 47th Congress, 1st Sess (1881) vol 12, pp 704-705.

after the 1900 census, prompting Maine representative John E. Littlefield to declare, "God help the State of Maine when mathematics reach for her and undertake to strike her down."[10] This bizarre phenomenon often occurs in Hamilton's method, and is largely responsible for the abandonment of the method by the U.S. Congress at the turn of the century. Another problem with Hamilton's method is the *population paradox*, in which a state that is growing quickly loses representatives to a state that is growing more slowly. Divisor methods avoid both the Alabama and population paradoxes, and are the only apportionment methods for which the population paradox never occurs.[11]

An additional paradox from which Hamilton's method suffers is the *new states paradox*. As the name suggests, this paradox arises when new states are added. For example, suppose a territory with a population of 604,642 is admitted to our fictitious country and becomes State J. Currently each representative has $4,688,782/100 \approx 46,888$ constituents, so State J should receive an apportionment of $604,642/46,888 \approx 13$ seats. Therefore, thirteen seats are added to the legislature, and one might expect that the apportionments of the other states remain the same. However, Table 5 shows that this is not the case.

Table 5: New States Paradox under Hamilton's Method

State	Population	Apportionment		Population	Apportionment
State E	163,904	3		163,904	4
State H	70,173	2		70,173	1
State J	—	—		604,642	13
Total	4,688,782	100		5,293,424	113

As in the case of the Alabama and population paradoxes, the new states paradox does not occur with the use of any of the divisor methods.[12]

All of the paradoxes I have discussed appear when Hamilton's method is used, but divisor methods also admit paradoxes. One example is the *migration paradox*,[13] an example of which is given in Table 6.

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10. Reapportionment Act H 12740, 56th Cong 2d Sess (Jan 5, 1901), in 34 Cong Rec H 593 (1901).

11. Balinski and Young, *Fair Representation* at 70 (cited in nite 2).

12. Id.

13. B.A. Bradberry, *A Geometric View of Some Apportionment Paradoxes*, 65 Mathematics Magazine 1, 16 (1992).

Table 6: Migration Paradox under Hill's Method

State	Population	Apportionment	Population	Apportionment
State A	1,908,578	40	1,908,578	41
State C	651,832	14	300,832	6
State G	120,419	3	471,419	10
Total	4,688,782	100	4,688,782	100

In Table 6, 351,000 people have migrated from State C to State G, while the populations of the other six states remain fixed. Of course, this migration affects the apportionments of States C and G, but observe that State A also gains a representative. The migration paradox is not just a defect of Hill's method; it can occur with the use of any apportionment method, including Hamilton's method.

WHAT APPORTIONMENT METHOD SHOULD WE USE?

As we have seen, each of the various methods that have been used to apportion the U. S. House of Representatives has unpleasant political and mathematical consequences. A natural question arises: is there an apportionment method that avoids these paradoxes? At the very least, it would be desirable to have a method that satisfies the following axioms:

Axioms for an Apportionment Method

1. *Population Monotonicity*: No state that gains population gives up a seat to one that loses population.

2. *House Monotonicity*: If each state's population remains fixed and the house size increases, then no state loses a seat.

3. *Quota*: No state's apportionment deviates from its quota by more than one seat.

These axioms appear quite minimal, and there are surely others that one could add to this list. However, the foregoing axioms are sufficient to cause problems; Balinski and Young proved that there exists no apportionment method satisfying these three axioms simultaneously. In fact, more is true—we cannot even find an apportionment method that is population monotone and does not

violate quota.[14] Therefore, we must be resigned to the fact that whatever apportionment method we choose, it will suffer from at least some of the paradoxes I have described.

There does not exist a perfect apportionment method, but what is the *best* apportionment method? To answer this question, it is necessary to define what "best" means. Certainly the method should be fair, but the meaning of "fair" in this context is subject to interpretation. For example, one might argue that an apportionment method is fair if it favors neither the large states nor the small states. On the other hand, since the composition of the U.S. Senate favors small states, it is not unreasonable to utilize an apportionment method such as Jefferson's, which would favor the large states in the U.S. House. Historically, most Representatives have sought apportionment methods that treat large and small states equally.

The problem of finding the best apportionment method has been a subject of intense debate for much of the history of the United States, and in the first part of the twentieth century, two groups of mathematicians weighed in with their opinions. In 1929, four prominent mathematicians who were members of the National Academy of Sciences wrote a report that favored Hill's method.[15] Largely on the basis of this report, Congress permanently adopted Hill's method in 1941.[16] In 1948, the eminent mathematicians Marston Morse, John von Neumann, and Luther Eisenhart were asked to examine the apportionment problem, and they concurred with the 1929 report.[17]

In spite of the impressive list of proponents of Hill's method, the reasoning used by these mathematicians was sophistic at best. The mathematicians who wrote the 1929 report considered only five divisor methods, which they called the method of smallest divisors (Adams' method), the method of greatest divisors (Jefferson's method), the method of major fractions (Webster's method), the method of equal proportions (Hill's method), and the method of the harmonic mean (Dean's method,[18] which was never seriously considered for adoption). The methods were ranked in order of the extent to which they

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14. M.L. Balinski and H. Peyton Young, *The Apportionment of Representation*, 33 Proceedings of Symposia in Applied Mathematics 1, 25 (1985).

15. G.A. Bliss, E.W. Brown, L.P. Eisenhart, R. Pearl, *Report to the President of the National Academy of Sciences* 21-3 (1929).

16. The Congress 2 USC §2a (1941); *US Statutes at Large*, 1941-42, vol 55, part 1, pp 761-762.

17. M. Morse, J.v. Neumann, L. Eisenhart, "Report to the President of the National Academy of Sciences" (May 28, 1948), mimeograph.

18. James Dean was professor of astronomy and mathematics at Dartmouth and the University of Vermont. The rounding rule that he suggested was to round up or down according to whether a state's quota was greater or less, respectively, than the harmonic mean of the floor and ceiling of the quota. The harmonic mean of two numbers is 2 divided by the sum of the reciprocals of the numbers. For more information, see Balinski and Young, *Fair Representation* at 29.

favored small states over large states, which put Hill's method in the middle. On the basis of this reasoning, the four mathematicians decreed that Hill's method must be the one that is unbiased. This reasoning is rather fatuous, since there are an infinite number of divisor methods, not just five. Certainly these mathematicians were capable of a more penetrating mathematical analysis of the problem than this; it seems that they did not take the problem very seriously. The 1948 report largely echoes the 1929 report (Eisenhart was one of the four who contributed to the 1929 report), so it appears that Morse, von Neumann, and Eisenhart also failed to give serious consideration to the problem.

Starting in the 1970's, several researchers began a careful mathematical study of apportionment methods, and Balinski and Young proved that among the apportionment methods we have considered in this Article, only Webster's method is free of bias.[19] Moreover, Webster's method, like all divisor methods, is both population monotone and house monotone. It is possible for Webster's method to violate quota, but it is the divisor method that is least likely to do so. Finally, Webster's method is the only divisor method that satisfies *nearness of quota*; an apportionment is near quota if it is not possible to transfer a seat from one state to another and bring both states closer to their quotas.[20] One can also argue that the rounding algorithm that Webster's method uses (rounding to the nearest whole number) intuitively seems the fairest. As Representative John A. Anderson put it in 1882,

> Since the world began there has been but one way of proportioning numbers, namely, by using a common divisor, by running the "remainders" into decimals, by taking fractions above .5, and dropping those below .5; nor can there be any other method. This process is purely arithmetical . . . If a hundred men were being torn from limb to limb, or a thousand babies being crushed, this process would have no more feeling in the matter than would an iceberg; because the science of mathematics has no more bowels of mercy than has a cast-iron dog.[21]

EPILOGUE: HILL VS. WEBSTER

As the foregoing examples suggest, there exists no perfect method of apportionment. Every method that has ever been used or will be used has at

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19. Balinski and Young, *The Apportionment of Representation* at 1-29 (cited in note 14). There are other divisor methods that are unbiased, but they involve bizarre rounding rules that are politically and aesthetically unpalatable; see id at 20.

20. *Fair Representation* at 82 (cited in note 2).

21. Apportionment of Representatives, H 3550, 47th Cong, 1st Sess (February 15, 1882), in 13 Cong Rec H 1179 (1882).

least one drawback that makes it politically and mathematically distasteful. Webster's method is the closest to being flawless, but it seems unlikely (at least for the foreseeable future) that Congress will reconsider its decision to use Hill's method. Nonetheless, there is a compelling argument that Webster's method is significantly superior to Hill's method.

As I mentioned above, Webster's method can violate quota, but it does so very rarely. A violation of quota occurs with Webster's method approximately once every 1640 apportionments, or 16,400 years.[22] Furthermore, if Webster's method had been used to apportion every session of the House of Representatives since 1792, not a single quota violation would have occurred.[23] Hill's method is also unlikely to violate quota; a quota violation in Hill's method should be expected only once every 3500 years.[24] Thus, quota violations are quite rare in either one of these apportionment methods, and if quota violation were the only issue, there would not be a compelling reason for switching from Hill's method to Webster's method.

The more serious drawback to Hill's method is that it is biased toward states with small populations. If one were to look at a large number of apportionments using Hill's method, one would find that per capita, small states average 3.5% more representatives than do large states.[25] There already exists a bias toward small states in the House, since every state, no matter how meagerly populated, is granted at least one representative. In addition, as I mentioned earlier, the Senate is also biased toward small states. All of these facts support Webster's method over Hill's.

Another advantage that Webster's method enjoys is that it tends to keep states' apportionments closer to their quotas than does Hill's method. For example, if Webster's method had been used to apportion the House following the 1920 census, six more states' apportionments would have been closer to quota than would have been the case with Hill's method.[26]

The final benefit of Webster's method that I mention here is its simplicity. The idea of rounding to the nearest whole number is easy to explain and easy to understand. In contrast, the "geometric mean ratio" rounding rule that Hill's method employs seems quite unnatural and is somewhat difficult to understand, especially to those who are not mathematicians. It would be interesting to poll the House of Representatives on this subject; I conjecture few Representatives even know that Hill's method is the current method of

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22. *Fair Representation* at 81 (cited in note 2).
23. Id at 82.
24. Id at 81.
25. Id at 77.
26. Id at 82.

apportionment, and I surmise that almost no one in the House could explain the details of the method. However, until some state decides that it is being unfairly treated, Hill's method will probably remain the law of the land.